Problem Set 10

DUE: Never.

1. [BRETSCHER, SEC. 7.1 #12] Find the eigenvalues and eigenvectors of $A := \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$.

2. [BRETSCHER, P. 318,SEC. 7.2 #28] [Problem on Markov Chains].

3. If $\vec{v}$ is an eigenvector of the matrix A, show that it is also an eigenvector of $A + 37I$. What is the corresponding eigenvalue?

4. Let A be an invertible matrix. Show that $\lambda = 0$ cannot be an eigenvalue.

   Conversely, if a (square) matrix is not invertible, show that $\lambda = 0$ is an eigenvalue.

5. Let $z = x + iy$ be a complex number. For which real numbers x, y is $|e^z| < 1$?

6. Let M be a 4×4 matrix of real numbers. If you know that both $1 + 2i$ and $2 - i$ are eigenvalues of M, is M diagonalizable? Proof or counterexample.

7. Let A and B be $n \times n$ real positive definite matrices and let $C := tA + (1 - t)B$. If $0 \le t \le 1$, show that C is also positive definite. [This is simple. No "theorems" are needed.]

8. Let $A := \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

   a) Find the eigenvalues and eigenvectors of A.

   b) Find an orthogonal transformation R so that $R^{-1}AR$ is a diagonal matrix.

9. If $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, solve $\dfrac{d\vec{x}}{dt} = A\vec{x}$ with initial condition $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

10. Let A and B be any 3×3 matrices. Show that $\operatorname{trace}(AB) = \operatorname{trace}(BA)$. [This is also true for $n \times n$ matrices.]

    Use this to give another proof that if the matrices M and Q are similar, then $\operatorname{trace}(M) = \operatorname{trace}(Q)$.

11. Let $A := \begin{pmatrix} 1/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix}$.

   a) Compute A^{50}.

   b) Let $P_0 := \begin{pmatrix} p \\ q \end{pmatrix}$ where $p > 0$ and $q > 0$ with $p + q = 1$. Compute $A^{50}P_0$. What do you suspect $\lim_{k\to\infty} A^k P_0 =?$.

   c) Note that A is the transition matrix of a Markov process. What do you suspect is the long-term stable state? Verify your suspicion.

12. Let A be a 3×3 matrix whose eigenvalues are $-1 \pm i$ and -2. If $\vec{x}(t)$ is a solution of $\dfrac{d\vec{x}}{dt} = A\vec{x}$, show that $\lim_{t\to\infty} \vec{x}(t) = 0$ independent of the initial value $\vec{x}(0)$.

13. a) If $B := \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$, find a self adjoint matrix Q so that $Q^2 = B$. [This should be obvious.]

   b) If $A := \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$, find a self adjoint matrix P so that $P^2 = A$.

14. Let $A := \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$. Solve $\dfrac{d^2\vec{x}(t)}{dt^2} + A\vec{x}(t) = 0$ with $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{x}\,'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

   [Remark: If A were the diagonal matrix $\begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$, then this problem would have been simple.]

15. Let A be an $n \times n$ matrix that commutes with *all* $n \times n$ matrices, so $AB = BA$ for all matrices B. Show that $A = cI$ for some scalar c. [Suggestion: Let $\vec{v}$ be an eigenvector of A with eigenvalue λ].

[Last revised: December 3, 2012]