Representing Symmetries by Matrices

If order to understand and work with the symmetries of an object (the symmetries of a square is a simple example), one would like a way to compute, not just wave your hands. For an object with complicated symmetries, this is essential. The standard technique goes back to Descartes' introduction of coordinates in geometry. Say one has two copies of the plane, the first with coordinates (x_1, x_2), the second with coordinates (y_1, y_2). Then the high school equations

$$x_1 + 2x_2 = y_1 \quad (1)$$
$$x_1 - x_2 = y_2 \quad (2)$$

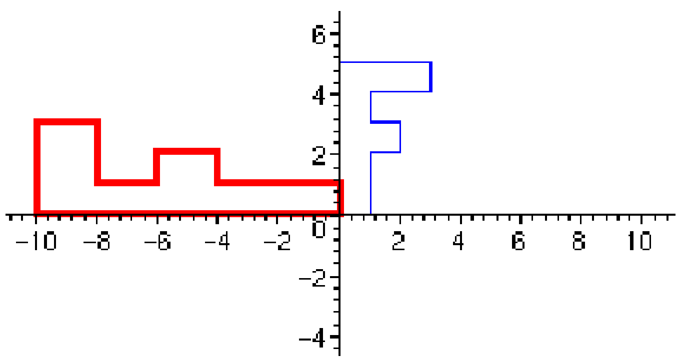
can be thought of as a mapping from the (x_1, x_2) plane to the (y_1, y_2) plane. For instance, if $x_1 = 1$ and $x_2 = -1$, then $y_1 = -1$ and $y_2 = 2$. Thus the point $(1, -1)$ is mapped to the point $(-1, 2)$.

Similarly, the equations

$$0x_1 - 2x_2 = y_1 \quad (3)$$
$$x_1 + 0x_2 = y_2 \quad (4)$$

defines a map that takes $(1, 0)$ to $(0, 1)$ and $(0, 1)$ to $(-2, 0)$. It can be thought of as a vertical stretching by a factor of 2 followed by a counter-clockwise rotation by 90 degrees.

Here is a picture of what this map does to the letter F: it maps the light F to the dark F.



The essence of equations (1)-(2) and (3)-(4) is in the coefficients of x_1 and x_2 and is captured concisely in the respective matrices

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}.$$

Similarly, the equations below define a *counterclockwise rotation*, R, by 90

$$\begin{aligned} -x_2 &= y_1 \\ x_1 &= y_2 \end{aligned} \qquad \text{with matrix} \qquad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

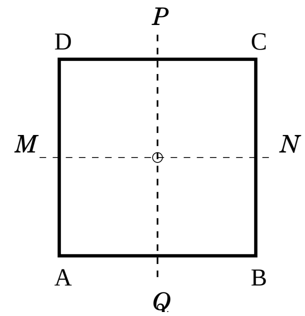
The following equations define a *reflection*, S, across the vertical axis:

$$\begin{aligned} -x_1 &= y_1 \\ x_2 &= y_2 \end{aligned} \qquad \text{with matrix} \qquad S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

so you reverse the sign of the first coordinate and leave the second coordinate unchanged.

Exercise Geometrically interpret the effect of the matrix $C := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Show that $B = RC$ and interpret this geometrically as the mapping C followed by the mapping R.

EXAMPLE To describe the symmetries of a square $ABCD$, introduce coordinates so that the center of the square is at the origin. One obvious symmetry is the 90 degree rotation R described above. Then R^2 (just repeat R) is the rotations by 180 degrees. Also R^3 is the rotation by 270 degrees – which is clearly equivalent to a *clockwise* rotation by 90 degrees, which we write as $R^{-1} = R^3$. A rotation by 360 degrees is the same as no rotation, so R^4 is the identity matrix: $R^4 = I$. Observe $R^{-1}R = R^3R = R^4 = I$, as one should want. Another evident symmetry is the reflection, S, across the vertical line PQ. Clearly reflecting twice brings you back home, so $S^2 = I$.



We can use a sequence of these symmetries, such as SR (a rotation R followed by a reflection S), to get the complete *group of symmetries of the square*. The complete list of elements of this group are:

$$I, \quad R, \quad R^2, \quad R^3, \quad S, \quad SR, \quad SR^2, \quad SR^3. \tag{5}$$

Note that by a computation, $S^2 = I$, $RS = SR^3$, $R^2S = SR^2$, and $R^3S = SR$ so the above list contains all possible combinations of products of R's and S's. Since $SR \neq RS$, this group of symmetries is *not* commutative.

There are some additional evident symmetries of the square, for example the reflection T across the horizontal line MN. Is this missing from our list (5)? If you sketch the figures, you will see that you can achieve T by first using the reflection S followed by R^2. Thus, $T = R^2S$. Similarly, the reflection across the diagonal DB is equivalent to RS. The list (5) really does contain *all* the symmetries of the square.

EXERCISE:

a) Use $RS = SR^3$ to show that the maps RSR, R^2S, and RSR^{-1} are in the list (5).

b) Prove that the list (5) really does contain *all* the symmetries of the square. I suggest beginning with the special case where the vertex A is fixed. What are the possible adjacent vertices? A key ingredient is that the symmetries of the square are *rigid motions*, that is, they preserve distances between points, so no stretching or shrinking is allowed.

These matrices (5) give a *representation* of the group of symmetries of a square. Their value is they are very specific and can be used for technical computations. While the machinery is excessive for something as simple as the symmetries of a square, it is vital to understand the symmetries of more complicated objects — such as sub-atomic particles. This is the subject of *group representation theory*.