Functions of Several Variables
Maxima and Minima

Functions of one variable (review).

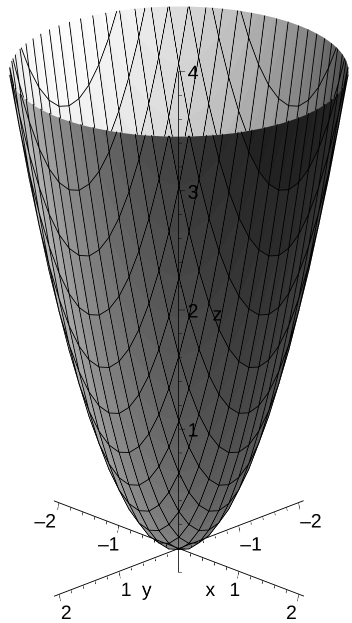
Interpret the function:

as a **graph** $y = f(x)$
as the **position of a particle** $y = g(t)$ at time t.

The derivative: **slope of tangent line** or **velocity**.

At a local maximum or minimum the derivative is zero.

EXAMPLE: **Standard minimum** $f(x,y) = x^2 + 3y^2$



Find critical points:

$$\partial_x f(x,y) = 2x, \qquad \partial_y f(x,y) = 6y$$

so the only critical point is the origin, $(0,0)$.

Second derivative test:

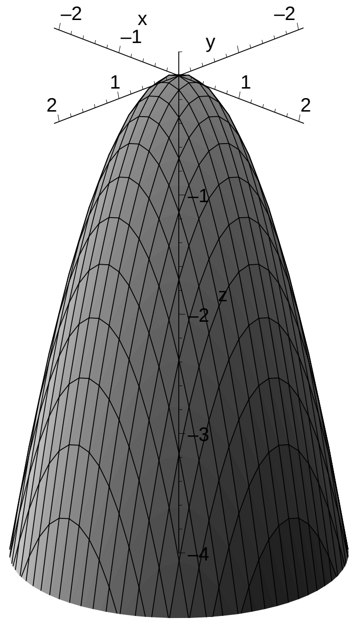
$$\partial_{xx} f(x,y) = 2, \qquad \partial_{xy} f(x,y) = 0, \qquad \partial_{yy} f(x,y) = 6$$

$f''(0,0)$ is the diagonal matrix

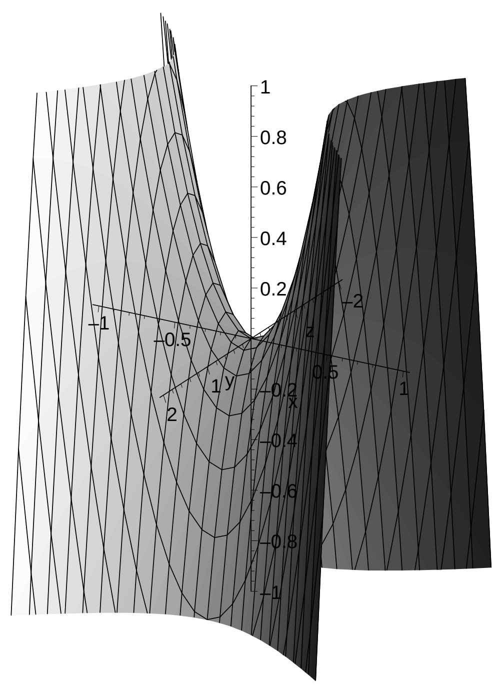
$$f''(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}.$$

This is *positive definite* so the origin is a local minimum.

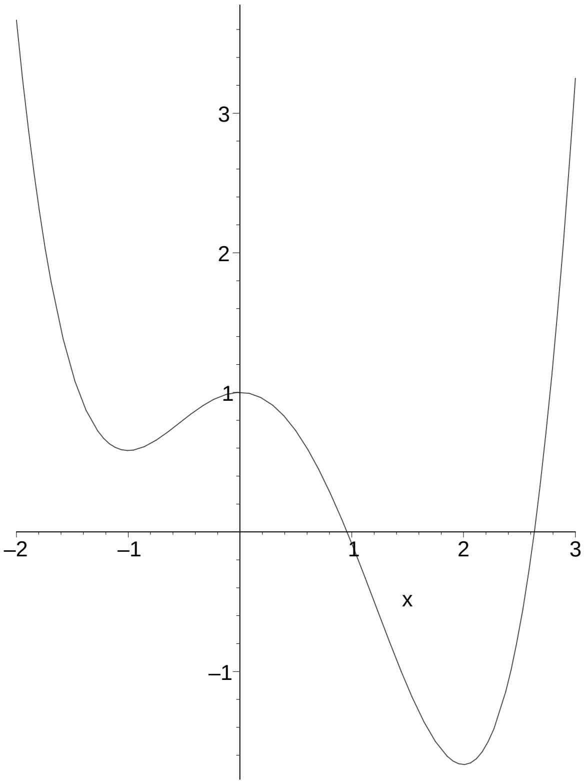
EXAMPLE: **Standard maximum** $f(x,y) = -(x^2 + y^2)$

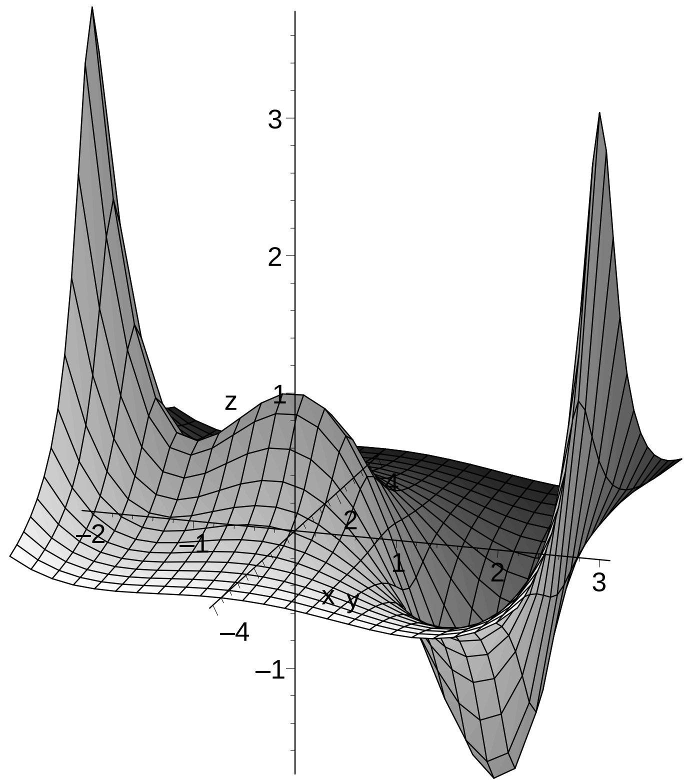


EXAMPLE: **Standard saddle** $f(x,y) = -x^2 + 3y^2$



EXAMPLE: $$f(x,y)=\frac{3x^4-4x^3-12x^2+12}{12(1+y^2)}$$





The curve on the left is $f(x,0)$. From the graph you see one saddle, one max, and one min, all on the x axis.

Compute the critical points:

$$\partial_x f(x,y)=\frac{x^3-x^2-2x}{1+y^2},\quad \partial_y f(x,y)=\frac{-(3x^4-4x^3-12x^2+12)y}{6(1+y^2)^2}$$

Critical points: $(0,0)$, $(-1,0)$, $(2,0)$.

Second derivative test. The second partial derivatives take more work to compute:

$$f''(x,y) = \begin{pmatrix} \frac{3x^2-2x-2}{1+y^2} & \frac{-2(x^3-x^2-2x)y}{(1+y^2)^2} \\ \frac{-2(x^3-x^2-2x)y}{(1+y^2)^2} & \frac{(3y^2-1)(3x^4-4x^3-12x^2+12)}{6(1+y^2)^3} \end{pmatrix}$$

Thus, the second derivative matrices at the critical points are:

$$f''(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \qquad \text{max}$$

$$f''(2,0) = \begin{pmatrix} 6 & 0 \\ 0 & \frac{10}{3} \end{pmatrix} \qquad \text{min}$$

$$f''(-1,0) = \begin{pmatrix} 3 & 0 \\ 0 & \frac{-7}{6} \end{pmatrix} \qquad \text{saddle}$$

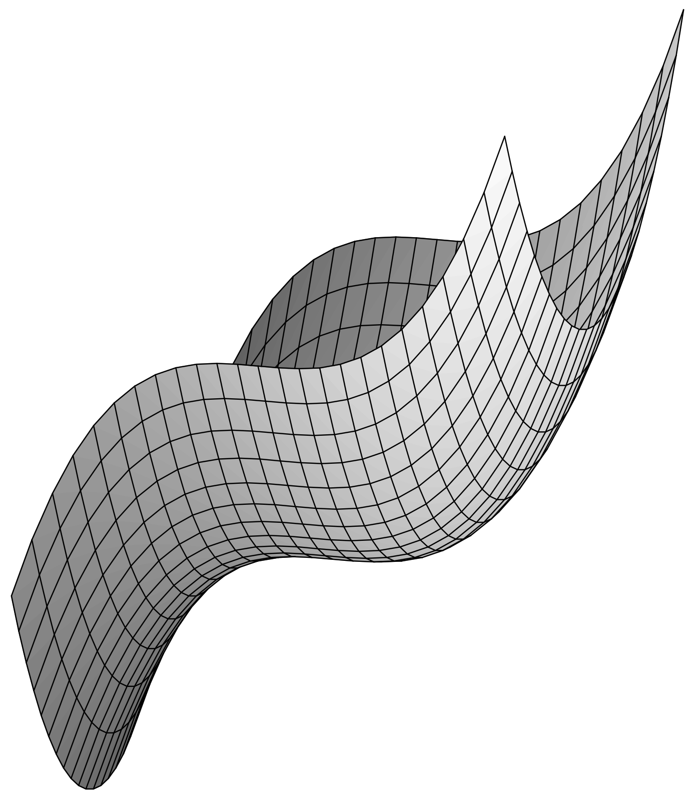
EXAMPLES OF DEGENERATE CRITICAL POINTS

Moral: the second derivative test is inconclusive.

Degenerate saddle at the origin:

$$f(x,y) = x^2 + y^3$$

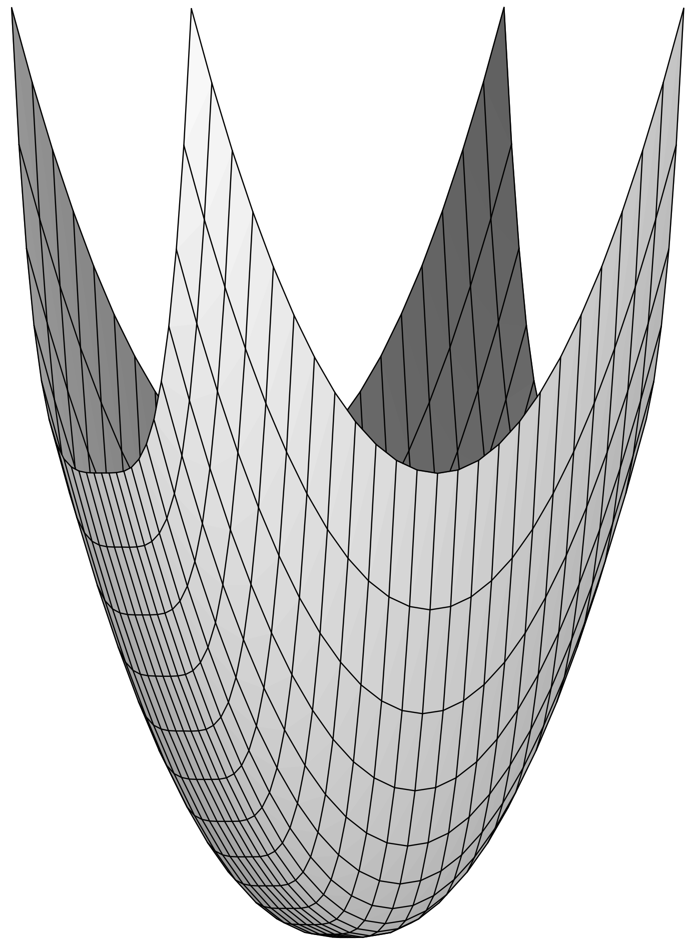
$$f''(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$



Degenerate minimum at the origin:

$$f(x,y) = x^2 + y^4$$

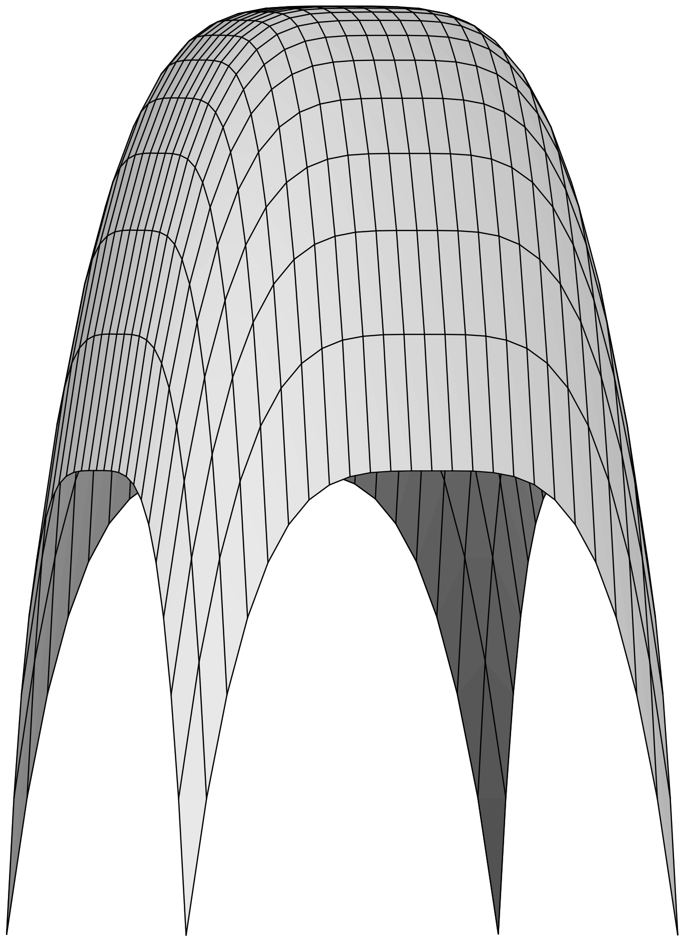
$$f''(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$



Degenerate maximum at the origin:

$$f(x,y) = -(x^4 + y^4)$$

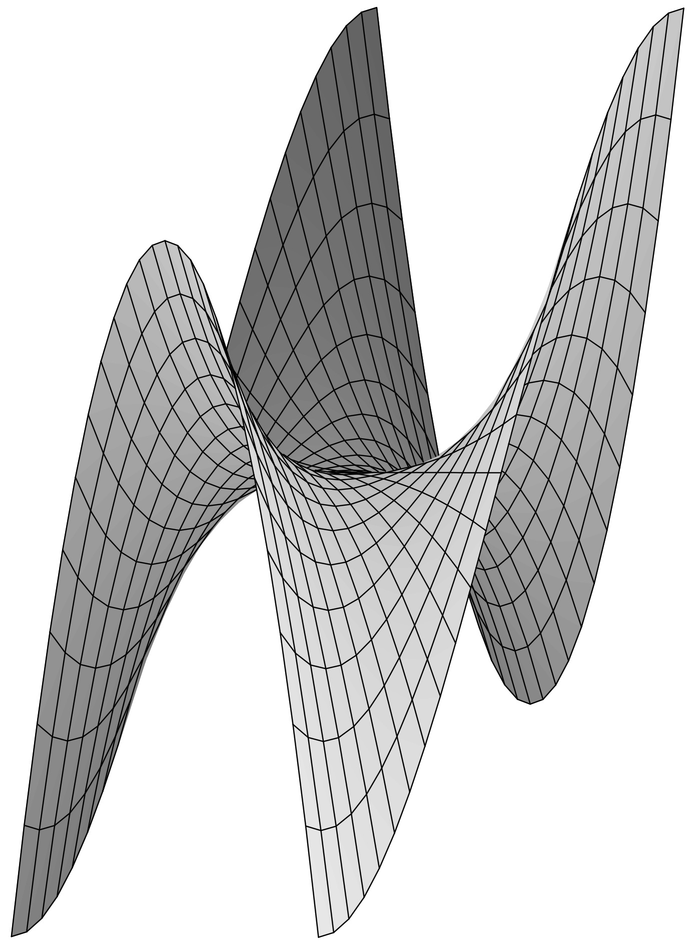
$$f''(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$



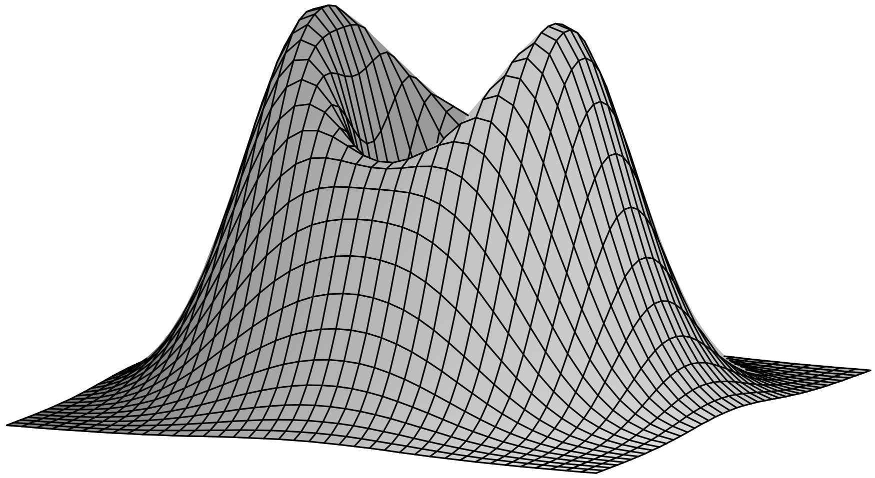
Degenerate *monkey* saddle at the origin:

$$f(x,y) = x^3 - 3xy^2 = \mathcal{R}\{(x+iy)^3\}$$

$$f''(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$



EXAMPLE: $f(x,y) = (2x^2 + 3y^2)e^{(1-x^2-y^2)}$



Clearly we see five critical points: two maxima, two saddle points, and one minima (in the pit).

Find them:

$$\partial_x f(x,y) = 2x[2 - (2x^2 + 3y^2)]e^{(1-x^2-y^2)}$$
$$\partial_y f(x,y) = 2y(3 - (2x^2 + 3y^2)]e^{(1-x^2-y^2)}.$$

So $\partial_x f(x,y) = 0$ and $\partial_y f(x,y) = 0$ at the five points

$$(0,0), \quad (\pm 1, 0), \quad \text{and} \quad (0, \pm 1).$$

Classify the critical points (second derivative test):

$$\partial_{xx} f(x,y) = 2[2-8x^2-(1-2x^2)(2x^2+3y^2)]e^{1-x^2-y^2}$$
$$\partial_{xy} f(x,y) = 4xy[-5+(2x^2+3y^2)]e^{1-x^2-y^2}$$
$$\partial_{xy} f(x,y) = 2[3-12y^2-(1-2y^2)(2x^2+3y^2)]e^{1-x^2-y^2}$$

Thus the second derivative (Hessian) matrices

$$f''(x,y) = \begin{pmatrix} \partial_{xx} f(x,y) & \partial_{xy} f(x,y) \\ \partial_{xy} f(x,y) & \partial_{yy} f(x,y) \end{pmatrix}$$

at these five critical points are (as anticipated)

$$f''(0,0) = \begin{pmatrix} 4e & 0 \\ 0 & 6e \end{pmatrix} \qquad \text{local minimum}$$
$$f''(\pm 1,0) = \begin{pmatrix} -8 & 0 \\ 0 & 2 \end{pmatrix} \qquad \text{saddles}$$
$$f''(0,\pm 1) = \begin{pmatrix} -2 & 0 \\ 0 & -12 \end{pmatrix} \qquad \text{maxima.}$$

Exercise Let A be an $n \times n$ real invertible symmetric matrix and $f(X) := \langle x, Ax \rangle e^{-\|X\|^2}$, $X \in \mathbb{R}^n$. Show that critical points of f are precisely the origin and the $\pm$unit eigenvectors of A. If the eigenvalues of A are distinct, there are $2n+1$ critical points. [The classification of these critical points is more complicated – but reasonable. For instance, it is clear that $f''(0) = 2A$.]

Let $f(x,y)$ be a smooth function on $\mathbb{R}^2$ with only one critical point: a strict local minimum at the origin.
Must this be the global minimum?

For a function of one variable, this must be the global min – but not for functions of several variables. The simplest example is probably the polynomial

$$f(x,y) := (1-y)^3x^2 + y^2$$

Perhaps easier to visualize are

$$f(x,y) := (1-y^2)^3x^2 + y^2 \quad \text{and} \quad g(x,y) := \frac{(1-y^2)^3x^2 + y^2}{(1+y^2)^3}$$