Some Applications of Linear Algebra

1. Given n linear equations in n unknowns how can you tell
   a) when a solution exists?
   b) if that solution is unique?

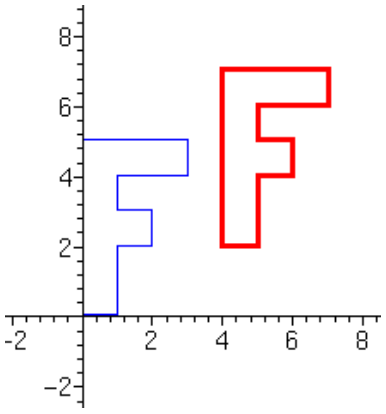
2. Linear maps $F(X) = AX$, where A is a matrix, have the property that $F(0) = A0 = 0$, so they necessarily leave the origin fixed. It is simple to extend this to include a translation,

$$F(X) = V + AX,$$

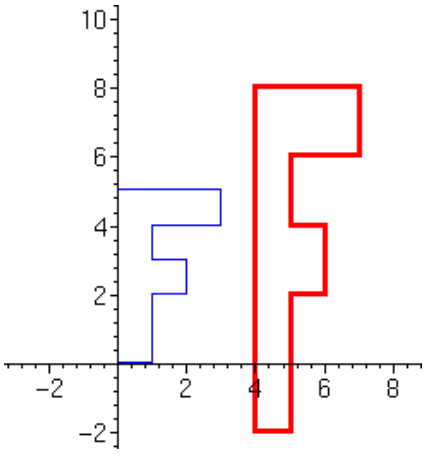
   where V is a vector. Note that $F(0) = V$.

   Find the vector V and the matrix A that describe each of the following mappings [here the light blue F is mapped to the dark red F].

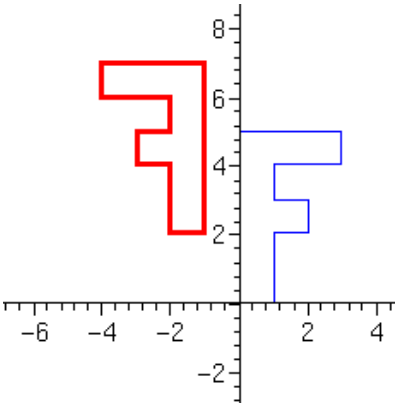
a).



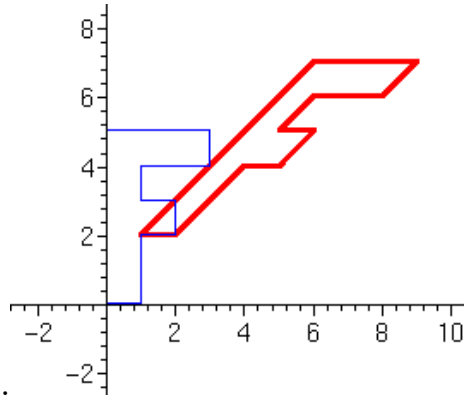
b).



c).



d).



3. There are two local branches of the Limousine Rental Company, one at the Airport and one in the City, as well as branches Elsewhere.

   Say every week of the limousines rented from the Airport 25% are returned to the City and 2% to branches located Elsewhere. Similarly of the limousines rented from the City 25% are returned to Airport and 2% to Elsewhere. Finally, say 10% of the limousines rented from Elsewhere are returned to the Airport and 10% to the City.

   If initially there are 35 limousines at the Airport, 35 in the City, and 150 Elsewhere, what is the long-term distribution of the limousines?

4. How does a Google search work?

5. a) Given a "tilted" ellipse such as $x^2 + 2xy + 2y^2 = 1$, how can you find orthogonal axes in which the ellipse is "untilted"?

   b) Evaluate $\iint_{\mathbb{R}^2} e^{-(x^2+2xy+2y^2)}\, dx\, dy$.

   c) Solve the coupled system of differential equations

$$\begin{aligned} x'(t) &= x + \ \ y \\ y'(t) &= x + 2y \end{aligned}$$

6. Given k points (x_j, y_j), $j = 1, \ldots, k$, in the plane with distinct x_j coordinates.

   a) Can you construct a polynomial that passes through all of the points?

   b) What is the "best" linear polynomial $y = ax + b$ that passes through these points? Estimate the error?

   c) Some experimental data (x_i, y_i) is believed to fit a curve of the form $y = (1 + x)/(a + bx^2)$, where the parameters a and b are to be determined from the data.

7. How can you find a polynomial (such as one stored on your calculator) that gives a good uniform approximation to the function $\sin x$ on the interval $-\pi/2 \leq x \leq \pi/2$?

8. What is the *second derivative test* for local maxima, minima, and saddles for a function $f(x, y)$ of several variables?