10

Vector Integrals and Stokes' Theorem

10.0 Introduction

This chapter is about "vector integration". For this new kind of integration, both the regions of integration and the integrands are of a different type from those considered in Chapter 9. We will learn to calculate integrals for which the region of integration is a curve in $\mathbb{R}^q$ (so-called "line integrals") and integrals for which the region of integration is a surface in $\mathbb{R}^3$ ("surface integrals"). The integrands will be of the form $\langle F(X), V(X)\rangle$, where F and V are vector-valued functions.

In this introductory section, we will describe briefly the kinds of curves and surfaces that we will use for regions of integration in our vector integrals. For line integrals, the region of integration will be a "smooth oriented curve" C. See Figure 10.1. The adjective "smooth" means that there is a tangent line at each point on the curve, except of course at its endpoints. The word "oriented" means that we will choose a direction along the curve, as indicated by the arrow in the figure. In surface integration, we will integrate over "smooth oriented surfaces" in $\mathbb{R}^3$, which are surfaces that have tangent planes at each point (except at the "surface boundary" or "edges"), and which are oriented by choosing one side as the "positive" side. For example, a sphere is a smooth surface which can be oriented by choosing either the inside or the outside as the positive side. It has no surface boundary. As a second example, consider a parallelogram region in $\mathbb{R}^3$, which is a smooth surface whose surface boundary consists of four line segments. If you think of this region as a thin wafer that is blue on one side and red on the other, you can orient it by choosing either the blue or the red side as its positive side.

As mentioned earlier, our integrands will be inner products of the form $\langle F(X), V(X)\rangle$, where F and V are vector fields defined on the region of integration. The vector field F may be thought of as a force field that surrounds the region of integration. See Figure 10.2, where F is indicated by the curved arrows ("streamlines"), and the region of integration is the surface $\mathcal{B}$, with surface boundary $\partial\mathcal{B}$. For each point X on the surface, the vector $V(X)$ is a special unit vector that indicates the orientation of the region of integration at X.

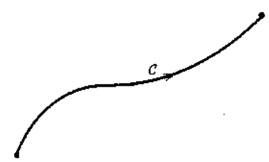

FIGURE 10.1. A curve in the plane

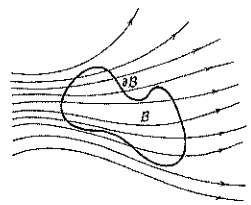

FIGURE 10.2. A vector field defined on $\mathcal{B}$

Line and surface integrals arise often in physics and engineering as well as in mathematics. In Section 10.1, we will discuss smooth oriented curves in more detail, and define the line integral. In Section 10.2, an important theorem (Stokes' Theorem) concerning line integrals is proved. It is a beautiful and useful generalization of the Fundamental Theorem of Calculus. In Section 10.3, smooth surfaces and surface integrals are treated, along with generalizations of Stokes' Theorem. We will see that both line and surface integrals can be calculated in terms of ordinary integrals (including double and triple integrals). The final section, Section 10.4, introduces an important type of vector field, called a "conservative force field", or a "potential field". Calculations of line integrals in conservative force fields are quite easy, since the answer only depends on the locations of the endpoints of the curve of integration. Throughout the chapter, there are discussions of applications of vector integration.

10.1 Line Integrals

10.1A Smooth Oriented Curves

Let $\mathcal{C}$ be a curve in $\mathbb{R}^q$. For example, see Figure 10.3, where a half circle in $\mathbb{R}^2$ is shown. The curve $\mathcal{C}$ may have two endpoints, as shown in the figure, or it may have one or no endpoints, for example, it may be a ray or a full circle. Let $\mathcal{C}_o$ be that part of the curve $\mathcal{C}$ that doesn't include the endpoints. Suppose for each point X of $\mathcal{C}_o$, there exists a *unit* vector $V = V(X)$ such that the line parametrized by $tV + X$ is the tangent line of $\mathcal{C}$ at X. We imagine that the unit vector $V(X)$ is attached to the curve at the point

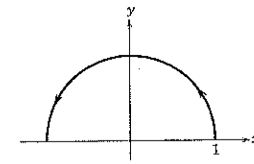

FIGURE 10.3. An oriented half circle

X. Note that $V : \mathcal{C}_o \to \mathbb{R}^q$ is a vector-valued function. If V is a continuous function, we say that $\mathcal{C}$ is a **smooth oriented curve** with **(positive) unit tangent vector** $V(X)$ at X. The direction of $V(X)$ is the orientation of the curve $\mathcal{C}$ at X. In Figure 10.3, the curve has been oriented so that it travels counterclockwise around the origin. You might imagine the positive tangent vectors attached to various points on the half circle in the picture. These tangent vectors point upward near the point $(1, 0)$, downward near $(-1, 0)$, and toward the left near $(0, 1)$.

Most often, we will work with curves in terms of parametrizations. A parametrization of a curve provides a natural orientation for that curve. If the curve $\mathcal{C}$ is parametrized by the function $\alpha : [a, b] \to \mathbb{R}^q$, and if $\alpha'(t)$ exists and is not $\mathbf{0}$ for $t \in (a, b)$, then a unit tangent vector at $X = \alpha(t)$ is given by

$$V(X) = \frac{\alpha'(t)}{\|\alpha'(t)\|} .$$

If $\alpha'(t)$ is continuous and not $\mathbf{0}$ at all $t \in (a, b)$, then $V(X)$ will also be continuous, and thus will provide an orientation for the curve. In this case, we will say that $\mathcal{C}$ is **oriented by** the parametrization α.

Examples:

1. Let $\alpha(t) = (\cos t, \sin t)$, $0 \le t \le \pi$. Then α parametrizes the curve in Figure 10.3. Since $\alpha'(t) = (-\sin t, \cos t)$ is a unit vector, we take $V(X) = \alpha'(t)$ for $X = \alpha(t)$. As you may easily verify, this parametrization provides the same orientation as that shown in the figure.
2. The half circle is also parametrized by $\alpha(t) = (\cos 2t, \sin 2t)$, $0 \le t \le \pi/2$. You should check that it also provides the same orientation.
3. The half circle in the figure is parametrized with the *opposite* orientation by $\alpha(t) = (\cos(\pi - t), \sin(\pi - t))$, $0 \le t \le \pi$.

Some curves, such as circles, have no endpoints. We call such curves **closed** curves. We will always insist in this chapter that if $\alpha : [a, b] \to \mathbb{R}^q$ is a parametrization of a closed curve $\mathcal{C}$ in $\mathbb{R}^q$, then $\alpha(a) = \alpha(b)$. The usual parametrization of the circle

$$\alpha(t) = (\cos t, \sin t)\,, \quad 0 \le t \le 2\pi$$

satisfies this condition.

10.1B DEFINITION OF THE LINE INTEGRAL

Let $\mathcal{C}$ be a smooth oriented curve in $\mathbb{R}^q$ with positive unit tangent vectors $V(X)$ for $X \in \mathcal{C}_o$. Also let $F : \mathcal{C} \to \mathbb{R}^q$ be a continuous vector field. Imagine that $\mathcal{C}$ has been partitioned into small pieces $\mathcal{C}_i$, and let X_i be a point in the piece $\mathcal{C}_i$. Let Δs_i be the arc length of the piece $\mathcal{C}_i$. Our approximation for the line integral of the vector field F along the oriented curve $\mathcal{C}$ is

$$\sum_i \langle F(X_i), V(X_i)\rangle \Delta s_i \,.$$

This formula has a somewhat more familiar appearance if we use the alternate notation

$$X \cdot Y = \langle X, Y\rangle$$

for the inner product of two vectors X and Y, in which case the synonym "dot product" is often used for "inner product". Then our approximating sum becomes

$$\sum_i F(X_i) \cdot V(X_i)\Delta s_i \,,$$

which looks a lot like an approximating Riemann sum. Under appropriate conditions, the approximating sums converge as the pieces $\mathcal{C}_i$ become smaller and more numerous to a quantity which we denote by

$$\int_{\mathcal{C}} F(X) \cdot V(X)\, ds = \int_{\mathcal{C}} F \cdot V\, ds \,.$$

This quantity is called the **line integral** of F along the oriented curve $\mathcal{C}$.

REMARK:

- There are several varations on the notation for the line integral. For example, we may write dX for $V\,ds$, leading to the notation

  $$\int_{\mathcal{C}} F \cdot V\, ds = \int_{\mathcal{C}} F(X) \cdot dX \,.$$

  Of course, dX is not really a vector, so $F \cdot dX$ is not defined. Nevertheless, this notation is well-established, and reminds us of the notation for ordinary single-variable integrals. It is also common to write the line integral in terms of components. For example, if $F = (p, q)$ is a vector field in $\mathbb{R}^2$, then we write

  $$\int_{\mathcal{C}} F(X) \cdot dX = \int_{\mathcal{C}} p(x, y)\, dx + q(x, y)\, dy \,,$$

  where we have expressed the "vector" X as $X = (dx, dy)$. We will switch among these various ways of writing the line integral, depending on which seems most convenient.

We need a way to calculate the line integral. This will be done in terms of a parametrization of the curve $\mathcal{C}$. Suppose that $\mathcal{C}$ is oriented by the parametrization α. Let the domain of α (an interval) be denoted by $\mathcal{R}$, and let $\mathcal{R}$ be partitioned into small intervals $\mathcal{R}_i$ centered at t_i and having arc length Δt_i. We know from our work with arc length (Section 4.3) that the arc length of the image of $\mathcal{R}_i$ under α is approximated by $\|\alpha'(t_i)\|\Delta t_i$. Let $X_i = \alpha(t_i)$. Since $V(X_i) = \alpha'(t_i)/\|\alpha'(t_i)\|$, the approximating sum for the line integral is itself approximated by

$$\sum_i F(X_i) \cdot \frac{\alpha'(t_i)}{\|\alpha'(t_i)\|}\|\alpha'(t_i)\|\Delta t_i = \sum_i F(\alpha(t_i)) \cdot \alpha'(t_i)\Delta t_i \,.$$

This last sum is an approximation for the (ordinary) integral

$$\int_{\mathcal{R}} F(\alpha(t)) \cdot \alpha'(t)\, dt \,.$$

Note that the integrand in this last integral is a *scalar*, since it is the inner product of two vector-valued functions. This reasoning justifies the following theorem:

Theorem 10.1.1 Let $\mathcal{C}$ be a smooth curve in $\mathbb{R}^q$ which is oriented by a parametrization α whose domain is an interval $[a, b]$, and let F be a continuous vector field whose domain includes $\mathcal{C}$. Then

$$\int_{\mathcal{C}} F \cdot V\, ds = \int_a^b F(\alpha(t)) \cdot \alpha'(t)\, dt \,.$$

As in our previous work with integration, we have not been completely rigorous in our definitions of line integrals, so it is not appropriate to try to give a proof of this theorem. There is however one aspect of the theorem that we wish to justify here. The right-hand side of the formula given in the theorem appears to depend on the parametrization α. We would like to know that if we use another parametrization $\beta : [c, d] \to \mathbb{R}^q$ that orients $\mathcal{C}$ with the same orientation, then

$$\int_a^b F(\alpha(t)) \cdot \alpha'(t)\, dt = \int_c^d F(\beta(s)) \cdot \beta'(t)\, ds \,.$$

(A similar situation occurred when we found a formula for arc length in Section 4.3c.) For simplicity, let us assume that $\beta = \alpha \circ \gamma$, where γ is an increasing function from $[c, d]$ to $[a, b]$. (It turns out that this assumption always holds if α' and β' are both never equal to $\mathbf{0}$.) Using the single-variable chain rule and the substitution $t = \gamma(s)$, we obtain

$$\begin{aligned}\int_c^d F(\beta(s)) \cdot \beta'(s)\, ds &= \int_c^d F(\alpha(\gamma(s))) \cdot \alpha'(\gamma(s))\gamma'(s)\, ds \\ &= \int_a^b F(\alpha(t)) \cdot \alpha'(t)\, dt \,.\end{aligned}$$

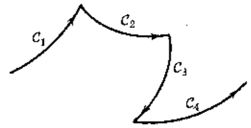

FIGURE 10.4. A piecewise smooth curve

EXAMPLE:

4. Let C be the oriented curve $\alpha(t) = (\cos 2t, \sin 2t)$ for $0 \le t \le \pi/2$. See Figure 10.3. (The image of α is the upper half-unit circle traversed counterclockwise.) Let us evalute

$$\int_C (x_2, x_1) \cdot dX ,$$

which we can also write as

$$\int_C x_2\, dx_1 - x_1\, dx_2 .$$

Since $\alpha'(t) = (-2\sin 2t, 2\cos 2t)$,

$$\int_C x_2\, dx_1 - x_1\, dx_2 = \int_0^{\pi/2} (-2\sin^2 2t - 2\cos^2 2t)\, dt = \int_0^{\pi/2} -2\, dt = -\pi.$$

There is an obvious way to extend the definition of the line integral to finite unions of smooth oriented curves. If we have two curves C_1 and C_2, then we write $C_1 + C_2$ for their union, and define

$$\int_C F(X) \cdot dX = \int_{C_1} F(X) \cdot dX + \int_{C_2} F(X) \cdot dX.$$

An important case in which it is appropriate to think of C as a union of smooth oriented curves is when C is a **piecewise smooth**. See Figure 10.4. Such curves have tangent lines except at a finite number of points along the curve (shown as "corners" in the illustration). We think of a piecewise curve C as a union of finitely many smooth curves, in the obvious way. A line integral along a piecewise smooth curve is calculated by integrating separately over each smooth piece, with each successive integration picking up where the previous one left off, and then adding the results. Incidently, the reason for our increased generality is not obscure. We merely want to include curves like triangles and squares.

EXAMPLE:

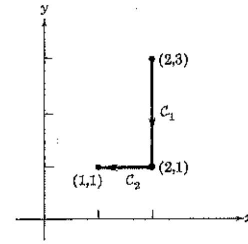

FIGURE 10.5. A curve composed of two line segments

5. Let $C = C_1 + C_2$, where C_1 is defined by $\alpha_1(t) = (2, 3-t)$ for $0 \le t \le 2$, and C_2 by $\alpha_2(t) = (4-t, 1)$ for $2 \le t \le 3$, and so C is composed of two straight-line segments. See Figure 10.5. Then, because

$$\int_C = \int_{C_1} + \int_{C_2},$$

we evaluate

$$\int_{C_1} x^2\, dx - xy\, dy = \int_0^2 2(3-t)\, dt = 8,$$

and

$$\int_{C_2} x^2\, dx - xy\, dy = \int_2^3 (4-t)^2(-dt) = -\frac{7}{3}.$$

Consequently

$$\int_C x^2\, dx - xy\, dy = 8 - \frac{7}{3} = \frac{17}{3}.$$

10.1C Properties of line integrals

The first and most basic property of a line integral is its linearity described in the following theorem.

Theorem 10.1.2 Let F and G be continuous vector fields, k a constant, and C a piecewise smooth oriented curve. Then the following properties hold:

1. $\int_C [F(X) + G(X)] \cdot dX = \int_C F(X) \cdot dX + \int_C G(X) \cdot dX$,
2. $\int_C kF \cdot dX = k \int_C F \cdot dX$, .

The next property is an inequality.

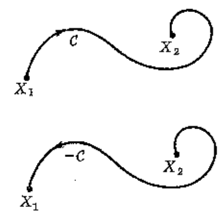

FIGURE 10.6. Two orientations of the same curve

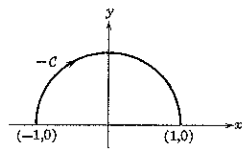

FIGURE 10.7. Reversing the orientation of the half circle

Theorem 10.1.3 Let F be a continuous vector field and let C be a piecewise smooth oriented curve in $\mathcal{D}(F)$. Assume that $\|F\| \le M$ for all X on the curve C. Then

$$\left|\int_C F(X) \cdot dX\right| \le ML\,,$$

where L is the arc length of C.

PROOF: Let α be a parametrization of the curve C and let $a < b$ denote the endpoints of $\mathcal{D}(\alpha)$. Then

$$\left|\int_C F(X) \cdot dX\right| = \left|\int_a^b F(\alpha(t)) \cdot \alpha'(t)\; dt\right| \le \int_a^b |F(\alpha(t)) \cdot \alpha'(t)|\; dt\,.$$

By the Cauchy-Schwarz inequality, we have

$$|F(\alpha(t)) \cdot \alpha'(t)| \le \|F(\alpha(t))\|\, \|\alpha'(t)\| \le M\|\alpha'(t)\|\,.$$

Therefore

$$\left|\int_C F(X) \cdot dX\right| \le M \int_a^b \|\alpha'(t)\|\; dt = ML\,.$$

Done. <<

If C is an oriented curve, then we denote by $-C$ the oriented curve that is obtained by reversing the orientation of C. See Example 3, illustrated by Figure 10.7, and also see Figure 10.6.

Theorem 10.1.4 If C is any piecewise smooth curve, then

$$\int_{-C} F(X) \cdot dX = -\int_C F(X) \cdot dX.$$

PROOF: This is essentially identical to the argument that the formula for the line integral does not depend on the parametrization, except that now $t = h(\tau) = a + b - \tau$, $a \le \tau \le b$, and $-C$ is parametrized as $\beta = \alpha \circ h$. The key change is in the limits of integration. Here $a = h(d)$ and $b = h(c)$. Thus

$$\begin{aligned}\int_C F(X) \cdot dX &= \int_d^c F(\beta(\tau)) \cdot \beta'(\tau)\; d\tau \\ &= -\int_c^d F(\beta(\tau)) \cdot \beta'(\tau)\; d\tau \\ &= -\int_{-C} F(X) \cdot dX.\end{aligned}$$

Done. <<

EXAMPLES:

6. One parametrization of the oriented line segment C from $(2, 3)$ to $(1, 1)$ shown in Figure 10.8 is

$$\alpha(t) = (2 - t, 3 - 2t)\,, \;\; 0 \le t \le 1\,.$$

(Note that the words "from $(2, 3)$ to $(1, 1)$" indicate the orientation.) Then

$$\int_C x^2\; dx - xy\; dy = \int_0^1 [-(2-t)^2 + 2(2-t)(3-2t)]\; dt = 4\,.$$

7. Let us evaluate

$$\int_C (2y - 3)\; dx + x^2\; dy\,,$$

where C is the curve from $(-1, 1)$ to $(1, 1)$ along the parabola $y = x^2$. We will use the parametrization

$$\beta(r) = (r, r^2)\,, \;\; -1 \le r \le 1\,.$$

We obtain

$$\int_C (2y - 3)\; dx + x^2\; dy = \int_{-1}^1 [(2r^2 - 3) + 2r^3]\; dr = -\frac{14}{3}\,.$$

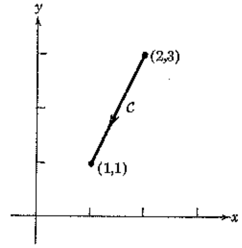

FIGURE 10.8. The line segment from $(2,3)$ to $(1,1)$

8. Let Γ denote the boundary of the triangular region with vertices at $(1,1)$, $(2,1)$, and $(2,3)$ traversed once counterclockwise. See Figure 10.9. We will compute

$$\int_\Gamma x^2\,dx - xy\,dy\,.$$

Again, this makes sense with any parametrization for Γ. Now $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$, where Γ_1 is the line segment from $(1,1)$ to $(2,1)$, Γ_2 is the line segment from $(2,1)$ to $(2,3)$, and Γ_3 is the line segment from $(2,3)$ to $(1,1)$. Thus $\Gamma_1 = -C_2$ and $\Gamma_2 = -C_1$ where C_1 and C_2 are the oriented curves in Example 5, and $\Gamma_3 = C$ the oriented curve in Example 6. Thus

$$\begin{aligned}\int_\Gamma &= \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} \\ &= \int_{-C_2} + \int_{-C_1} + \int_C = -\int_{C_2} - \int_{C_1} + \int_C \\ &= \frac{7}{3} - 8 + 4 = -\frac{5}{3}.\end{aligned}$$

Our calculations show that the line integrals along two different curves $\Gamma_1 + \Gamma_2$ and $-\Gamma_3$ from $(1,1)$ to $(2,3)$ are *different even though* the oriented curves begin and end at the same places; the values along the curves are $\frac{7}{3} - 8 = -\frac{17}{3}$ and -4, respectively. (This does *not* mean that different parametrizations of the same oriented curve would give different results.)

9. Let C denote the line segment from $(2,1)$ to $(1,1)$. One convenient parametrization is $\alpha(s) = (-s,1)$, $-2 \le s \le -1$. Thus

$$\int_C x^2\,dx - xy\,dy = \int_{-2}^{-1} -s^2\,ds = -\frac{7}{3}.$$

Of course, this should have been anticipated, because α is just a reparametrization of the curve C_2 in Example 2. Another method

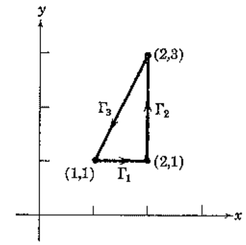

FIGURE 10.9. Traversing the boundary of a triangular region

for the same problem is to observe that $-C$ is somewhat simpler to parametrize: $\beta(t) = (t,1)$, $1 \le t \le 2$. Then we find that

$$\int_C = -\int_{-C} = -\int_1^2 t^2\,dt = -\frac{7}{3}.$$

10.1D Work

If the vector field F is a force field so that $F(X)$ is the force "felt" by a particle at the point X, then the line integral

$$\int_C F \cdot V\,ds$$

is defined in physics to be the **work** done by the force in moving the particle along the curve C. Since V is a unit vector, the dot product $F \cdot V$ is the length of the component of F in the direction of V, so only the component of the vector field F *tangent* to the curve influences the value of the line integral, and hence the work done by the force. In particular, if a force field is perpendicular to the curve C traveled by a moving particle, then this force does zero work.

For example, if the force field, like gravity, is vertical, then this force does no work in any horizontal motion of the object. Work done by this force is nonzero only when moving an object up or down. It is friction, not gravity directly, that causes us to expend work in sliding objects horizontally. On smooth ice, where sliding friction is quite small, it is very easy to move objects horizontally.

If the curve C is parametrized by a function $\alpha : [a,b] \to \mathbb{R}^q$, then the formula for the work can be written as

$$\text{work} = \int_a^b F(\alpha(t)) \cdot \alpha'(t)\,dt\,.$$

Since the velocity $\alpha'(t)$ appears in the formula for work, one might be led to believe that work depends on the velocity, and so if $\beta(\tau)$ describes the motion of another particle moving in the same direction along the same curve, then the work done by the force would be different for the two particle motions $\alpha(t)$ and $\beta(\tau)$. Not true. *The work does not depend on the particular motions but only on the curve and the direction traveled along the curve (that is, the orientation of the motion).* More precisely, we are just reasserting in a physical context the fact discussed earlier that the value obtained from the formula for a line integral does not depend on the parametrization, as long as the orientation remains the same. We can interpret Theorem 10.1.4 as stating that if a force F does work W in moving a particle along a curve C, then in running the particle backward, that is, along $-C$, the force does work $-W$.

Exercises

1. Compute $\int_C 2x\,dx + 6(x-y)\,dy$ where C is the oriented curve parametrized by
   (a) $\alpha(t) = (t,t)$, $0 \le t \le 1$
   (b) $\alpha(t) = (t,t^2)$, $0 \le t \le 1$
   (c) $\beta(s) = (\sin s, s)$, $0 \le s \le 2\pi$
   (d) $\Phi(t) = (e^t, 1)$, $0 \le t \le 1$;
   (e) $\alpha(\theta) = (1-\cos\theta, \sin\theta)$, $0 \le \theta \le \pi/2$
   (f) $\Psi(r) = (e^r, e^{-r})$, $-1 \le r \le 1$.
2. Compute $\int_C (x_2, x_3, x_1)\cdot dX$, where C is the oriented curve parametrized by
   (a) $\alpha(t) = (t,t,t)$, $0 \le t \le 1$
   (b) $\alpha(t) = tZ$, $0 \le t \le 1$, where Z is a fixed, but arbitrary member of $\mathbb{R}^3$
   (c) $(\cos t, \sin t, t)$, $0 \le t \le 2\pi$
   (d) $(\cos 2t \sin t, \sin 2t \sin t, \cos t)$
3. For the oriented curve C parametrized by
$$\alpha(t) = (\cos^2 t,\ \cos t \sin t,\ \cos t \sin t,\ \sin^2 t),\ 0 \le t \le \pi,$$
   compute $\int_C F(X)\cdot dX$, where
   (a) $F(X) = X$
   (b) $F(X) = X + (1,1,0,0)$
   (c) $F(X) = (x_1, -x_2, -x_3, x_4)$
   (d) $F(X) = (x_1, x_2, -x_3, -x_4)$.
4. Compute $\int_C 2x\,dx + 6(x-y)\,dy$ where
   (a) C is the graph of $y = 2x$ for $1 \le x \le 2$ and oriented in the direction of increasing x
   (b) C is the line segment beginning at $(-1,1)$ and ending at $(2,-2)$
   (c) C is the shortest curve from $(0,0)$ to $(1,1)$ that passes through the point $(1,0)$
   (d) C is the shortest curve from $(0,0)$ to $(1,1)$ that passes through the point $(0,1)$
   (e) C follows the parabola $x = y^2$, from $(1,-1)$ to $(1,1)$
   (f) C is the larger arc of the circle $(x-1)^2 + y^2 = 1$ from $(0,0)$ to $(1,1)$.
5. Compute $\int_C (6x - y^2)\,dx - 2xy\,dy$ for the curves C in parts (a) and (b) of Exercise 1 and parts (c) and (d) of Exercises 4. Compare your four answers.
6. Compute $\int_C X\cdot dX$, where C is a circle of radius $r \le 1$ on $S((0,0,0);1)$.
7. Evaluate $\int_C y^2\,dx + 3x^2\,dy$ for each of the following curves:
   (a) C is the line segment from $(-1,1)$ to $(2,-2)$,
   (b) C is line segment from $(2,-2)$ to $(-1,1)$.
8. Use answers to parts (a) of Exercise 1 and (d) of Exercise 4 to calculate
$$\int_C 2x\,dx + 6(x-y)\,dy,$$
   where C is the counterclockwise oriented boundary of the triangular region with vertices at $(0,0)$, $(0,1)$, and $(1,1)$.
9. Use parts (e) of Exercise 1 and (f) of Exercise 4 to calculate
$$\int_C 2x\,dx + 6(x-y)\,dy,$$
   where C is the entire circle $(x-1)^2 + y^2 = 1$ oriented counterclockwise.
10. Evaluate
$$\int_C e^x \sin y\,dx + e^x \cos y\,dy,$$
   where C is the portion of the circle $x^2+y^2 = 1$ with $x \ge 0$, oriented so that it begins at $(0,1)$ and ends at $(0,-1)$. *Hint:* In calculating the eventual single-variable integral avoid the tempatation to represent the integral of a sum as the sum of the integrals.
11. Evaluate $\int_C (x_1^2 + x_2^2)\,dx_1 + 2x_1x_2\,dx_2$, where C is the oriented curve represented parametrically as $(1-t^2, 3-2t)$ for $1 \le t \le 2$.
12. Let a force field be $F(x,y) = (xy, x^2 - 3y^2)$. Compute the work done by this force to move a particle of unit mass from $(0,0)$ to $(1,1)$ along each of the oriented curves in Exercise 1, parts (a) and (b).
13. Let a force field be $F(x,y) = (xy, x^2 - 3y^2)$. Compute the work done by this force to move a particle of unit mass from $(0,0)$ to $(1,1)$ along each of the oriented curves in Exercise 4, parts (c) and (d).
14. If a force field is $F(x,y) = (ye^{xy}, xe^{xy})$, find the work done by this force to move a particle of mass 1 counterclockwise around the boundary of the square $|x| \le 1$, $|y| \le 1$.

15. Here is a review problem. Let α be a differentiable parametrized curve whose image does not include $\mathbf{0}$, and let h be a scalar-valued differentiable function on the interval $(0, \infty)$. Show that

$$\frac{d}{dt}h(\|\alpha(t)\|) = \frac{h'(\|\alpha(t)\|)}{\|\alpha(t)\|}\alpha(t) \cdot \alpha'(t) .$$

16. Let C be a smooth curve in $\mathbb{R}^n$ with initial point U_0 and terminal point U_1. Assume that $\mathbf{0} \notin C$. Use the preceding exercise to show that

$$\int_C \frac{X}{\|X\|^3} \cdot dX = \frac{1}{\|U_0\|} - \frac{1}{\|U_1\|} .$$

17. Generalize the preceding exercise by replacing $\|X\|^3$ on the left-hand side by an arbitrary power of $\|X\|$ and changing the right-hand side appropriately.

18. For C, U_0, and U_1 as in Exercise 16 and ψ an arbitrary continuous scalar-valued function on the interval $(0, \infty)$, find a formula for

$$\int_C (\psi(\|X\|)X) \cdot dX .$$

Hint: To express your answer you will need to introduce a function not explicitly mentioned in the preceding sentence.

19. For each of the following statements decide whether it is true or false. If it is false, give a counterexample. If it is true, give a proof. Here F is a continuous vector field whose domain contains the smooth curve C.

    (a) If C is a vertical line segment in $\mathbb{R}^2$ and f_2 is the zero function, then $\int_C F(X) \cdot dX = 0$.

    (b) If C is a circle in the x_2x_3-plane in $\mathbb{R}^3$ and f_2 and f_3 are the zero function, then $\int_C F(X) \cdot dX = 0$.

    (c) If C is a circle in the x_2x_3-plane in $\mathbb{R}^3$ and f_2 and f_3 are negatives of each other, then $\int_C F(X) \cdot dX = 0$.

    (d) If each f_j is nonnegative, then $\int_C F(X) \cdot dX \geq 0$.

20. Let C denote the quarter-circle $x^2 + y^2 = 1$ in the first quadrant oriented counterclockwise.

    (a) Show that

$$\int_C -e^{(xy)^2}\,dx + e^{-(xy)^2}\,dy \leq \frac{\pi}{2}\sqrt{e^{1/16} + 1} .$$

    (b) Show that

$$\int_C -e^{(xy)^2}\,dx + e^{-(xy)^2}\,dy \leq \frac{\pi}{2}\sqrt{e^{1/16} + e^{-1/16}} .$$

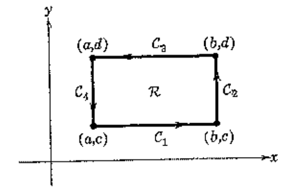

FIGURE 10.10. Stokes' Theorem on a rectangular region

21. Let $F(X)$ be a continuous force field in the plane. Suppose that a particle of mass m moves so that its position $\alpha(t)$ is determined by Newton's second law:

$$m\alpha'' = F \circ \alpha .$$

Show that the work done by the force F during the time interval $t_1 \leq t \leq t_2$ is equal to the change in kinetic energy of the particle:

$$\int_{t_1}^{t_2} (F \cdot \alpha')(t)\,dt = \frac{m}{2}\|\alpha'(t_2)\|^2 - \frac{m}{2}\|\alpha'(t_1)\|^2 .$$

Hint: First show that $\frac{1}{2}m(d/dt)\|\alpha'(t)\|^2 = (F \cdot \alpha')(t)$.

10.2 Stokes' Theorem in the Plane

10.2A Stoke's Theorem for a Rectangular Region

The time has come to generalize the Fundamental Theorem of Calculus to multiple integrals. Throughout this section, we will use the following notation: if $\mathcal{R}$ is a pathwise connected open set in $\mathbb{R}^2$, then we will denote the boundary of $\mathcal{R}$ by $\partial\mathcal{R}$. Here is the simplest case.

Theorem 10.2.1 Let $\mathcal{R}$ denote the rectangle $a < x < b$, $c < y < d$ and $F = (p, q)$ a vector field with a continuous derivative. Then

$$\int\int_{\mathcal{R}} \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\,dA = \int_{\partial\mathcal{R}} p\,dx + q\,dy,$$

where the boundary $\partial\mathcal{R}$ is oriented counterclockwise.

Proof: We treat the p and q terms separately. Now

$$\int\int_{\mathcal{R}} \frac{\partial q}{\partial x}\,dA = \int_c^d \left(\int_a^b \frac{\partial q}{\partial x}(x, y)\,dx\right)\,dy = \int_c^d [q(b, y) - q(a, y)]\,dy.$$

In the notation of Figure 10.10, however, we have

$$\int_c^d q(b, y)\,dy = \int_{C_2} q\,dy$$

and

$$\int_c^d q(a,y)\,dy = -\int_{-\mathcal{C}_4} q\,dy = \int_{\mathcal{C}_4} q\,dy,$$

and since $y \equiv$ constant on $\mathcal{C}_1$ and $\mathcal{C}_3$,

$$\int_{\mathcal{C}_1} q\,dy = 0 \quad\text{and}\quad \int_{\mathcal{C}_3} q\,dy = 0.$$

Therefore

$$\int\int_{\mathcal{R}} \frac{\partial q}{\partial x}\,dA = \int_{\partial\mathcal{R}} q\,dy.$$

Similar considerations give

$$\int\int_{\mathcal{R}} \frac{\partial p}{\partial y}\,dA = -\int_{\partial\mathcal{R}} p\,dx.$$

Adding these two equations, we arrive at the result. <<

10.2B More general regions

Precisely the same result holds for more general regions than rectangles. We prove this in two steps. First is the case where a region $\mathcal{B} \subseteq \mathbb{R}^2$ can be written both as

$$\varphi(x) \le y \le \psi(x), \qquad a \le x \le b,$$

so that φ and ψ are the lower and upper boundary curves, respectively, and as

$$\Phi(y) \le x \le \Psi(y), \qquad c \le y \le d,$$

so that Φ and Ψ are the left-hand and right-hand boundary curves, respectively. See Figure 10.11. We also suppose that the boundary is a piecewise smooth curve; that is, the boundary $\partial\mathcal{B}$ consists of a finite number of smooth curves. Such regions are called **simple regions**.

Theorem 10.2.2 Let $\mathcal{B} \subseteq \mathbb{R}^2$ be a simple region and $F = (p,q)$ a vector field with a continuous derivative. Then

$$\int\int_{\mathcal{B}} \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\,dA = \int_{\partial\mathcal{B}} p\,dx + q\,dy,$$

where the boundary $\partial\mathcal{B}$ is oriented counterclockwise.

Proof: We discuss only the p term. The q term is treated similarly. Now

$$\begin{aligned}\int\int_{\mathcal{B}} \frac{\partial p}{\partial y}\,dA &= \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} \frac{\partial p}{\partial y}(x,y)\,dy\right)\,dx \\ &= \int_a^b [p(x,\psi(x)) - p(x,\varphi(x))]\,dx.\end{aligned}$$

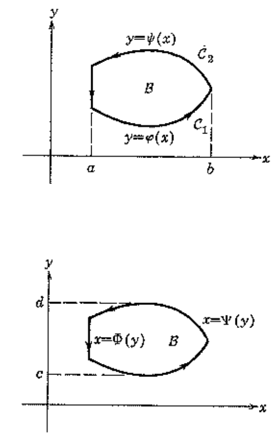

FIGURE 10.11. Stokes' Theorem on a region with curved sides

Let $\mathcal{C}_1$ denote the bottom curve of $\mathcal{B}$, and $\mathcal{C}_2$ the top curve, oriented so that $\partial\mathcal{B}$ is oriented counterclockwise. Then

$$\int_a^b p(x,\psi(x))\,dx = \int_{-\mathcal{C}_2} p\,dx, \qquad \int_a^b p(x,\varphi(x))\,dx = \int_{\mathcal{C}_1} p\,dx.$$

Thus, as claimed,

$$\int\int_{\mathcal{B}} \frac{\partial p}{\partial y}\,dA = -\int_{\mathcal{C}_1+\mathcal{C}_2} p\,dx = -\int_{\partial\mathcal{B}} p\,dx$$

(observe that $\int p\,dx = 0$ over any vertical line segment). <<

Finally, we extend this to more general regions, possibly containing holes. The idea is to dissect the region into simple regions and apply Theorem 10.2.2 to these simple regions separately. An example makes this clear. Let $\mathcal{B} \subseteq \mathbb{R}^2$ be the region indicated in Figure 10.12. We have dissected $\mathcal{B}$ into a number of simple regions. If one applies Stokes' Theorem to each of these simple regions separately, then line integrals along the interior dotted lines appear. The integrals along the dotted lines cancel, however, since, for example, the line shared by $\mathcal{B}_1$ and $\mathcal{B}_2$ is traversed once in each direction. By Theorem 10.1.4, the net result is zero. Thus, only integration along $\partial\mathcal{B}$ remains, that along the "outer" portion of $\partial\mathcal{B}$ is traversed counterclockwise and that along the "inner" portion is traversed clockwise. There is a mnemonic device that enables us to remember the orientation of the boundary: if you walk along $\partial\mathcal{B}$ so that your left hand is in $\mathcal{B}$, then you are walking in the positive direction of $\partial\mathcal{B}$. We call this the **positive orientation** of $\partial\mathcal{B}$.

The discussion above has established

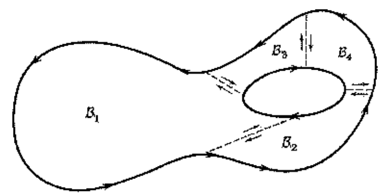

FIGURE 10.12. Dissecting $\mathcal{B}$ into simpler regions

Theorem 10.2.3 (Stokes) If $\mathcal{B} \subseteq \mathbb{R}^2$ is decomposable into a finite number of simple regions with piecewise smooth boundaries, then for any vector field $F = (p, q)$ with a continuous derivative,

$$\int\int_{\mathcal{B}} \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) \, dA = \int_{\partial\mathcal{B}} p(x,y) \, dx + q(x,y) \, dy,$$

where $\partial\mathcal{B}$ has positive orientation.

EXAMPLES:

1. Let $\mathcal{B}$ denote the triangular region with vertices at $(0,0)$, $(1,0)$, and $(0,1)$, with positively oriented boundary. We evaluate

$$\int_{\partial\mathcal{B}} (e^x + y - 2x) \, dx + (7x - \sin y) \, dy.$$

Let $p = e^y + y - 2x$ and $q = 7x - \sin y$. Then, by Stokes' Theorem,

$$\int_{\partial\mathcal{B}} (e^x + y - 2x) \, dx + (7x - \sin y) \, dy = \int\int_{\mathcal{B}} 6 \, dA.$$

But

$$\int\int_{\mathcal{B}} dA = \text{ area } (\mathcal{B}) = \frac{1}{2}.$$

Thus

$$\int_{\partial b} (e^y + y - 2x) \, dx + (7x - \sin y) \, dy = 3.$$

The theorem greatly simplified the amount of computation.

2. Let $\mathcal{B}$ denote the disk of radius a centered at the origin and let $p(r) = p(x,y)$, $q(r) = q(x,y)$ be functions that depend only on the distance r from the origin. Assume that p and q have continuous derivatives. Then by Stokes' Theorem

$$\int\int_{\mathcal{B}} \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) \, dA = \int_{\partial\mathcal{B}} p \, dx + q \, dy \, .$$

But on $\partial\mathcal{B} =$ circle of radius a, $p = p(a)$, $q = q(a)$ are constant. Therefore

$$\int\int_{\mathcal{B}} (q_x - p_y) \, dA = \int_{\partial\mathcal{B}} p(a) \, dx + q(a) \, dy.$$

To evaluate the line integrals, we could introduce a parametrization and go through the easy computation. Instead, we are sneaky and again apply Stokes' Theorem, using the fact that $p(a)$ and $q(a)$ are constants. A moment's thought (do not write anything) reveals that the value of the integral is zero; that is, if p and q are constant on the circle,

$$\int\int_{\mathcal{B}} (q_x - p_y) \, dA = 0.$$

10.2C Other Versions

In practice, people often use several different variants of Stokes' Theorem. For the first variant we need the idea of the **unit outer normal** vector to the boundary of a region $\mathcal{B}$. If $V(X)$ is the positive unit tangent vector at a point X on $\partial\mathcal{B}$, then the unit outer normal vector $N(X)$ at X is the unit vector obtained by rotating $V(X)$ 90 degrees *clockwise* in $\mathbb{R}^2$. In other words, $N(X)$ is perpendicular to $V(X)$, and hence perpendicular to the curve $\partial\mathcal{B}$. Furthermore, if you are facing in the direction of $V(X)$, then $N(X)$ will point toward your right, which is "outward" or away from the region $\mathcal{B}$. If $X = \alpha(t) = (\alpha_1(t), \ \alpha_2(t))$ is a parametrization of $\partial\mathcal{B}$, oriented positively, then

$$\frac{\alpha'(t)}{\|\alpha'(t)\|} = \frac{(\alpha_1'(t), \alpha_2'(t))}{\|(\alpha_1'(t), \alpha_2'(t))\|}$$

equals $V(X)$. If we apply the linear transformation with matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

to this vector (thus rotating it 90 degrees clockwise), we obtain

$$N(X) = \frac{(\alpha_2'(t), -\alpha_1'(t))}{\|(\alpha_2'(t), \alpha_1'(t))\|} \, .$$

EXAMPLE:

3. For a simple example of a unit outer normal vector, let $\mathcal{B}$ be the disk of radius 2 centered at the origin, so that $\partial\mathcal{B}$ is the circle of radius 2. See Figure 10.14. At a point (x,y) on $\partial\mathcal{B}$, the unit outer normal is given by $N = \frac{1}{2}(x,y) = (\cos t, \sin t)$, where $\partial\mathcal{B}$ is parametrized by $\alpha(t) = (2\cos t, 2\sin t)$. Note here that the unit tangent is $(-\sin t, \cos t) = \frac{1}{2}(-y, x)$.

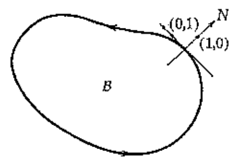

FIGURE 10.13. The outer normal vector N

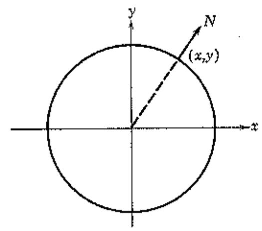

FIGURE 10.14. Stokes' Theorem on a circle

Let us now state the **divergence form** of Stokes' Theorem.

Theorem 10.2.4 (Divergence Theorem in $\mathbb{R}^2$) If $F = (f_1, f_2)$ is a vector field with a continuous derivative, and if $\mathcal{B} \subseteq \mathbb{R}^2$ satisfies the assumptions of Theorem 10.2.2, then

$$\int\int_{\mathcal{B}} \text{div } F \, dA = \int\int_{\mathcal{B}} \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right) dA = \int_{\partial\mathcal{B}} F \cdot N \, ds,$$

where N is the unit outer normal to $\partial\mathcal{B}$ and s is arc length on $\partial\mathcal{B}$.

PROOF: The first equality is the definition of the divergence:

$$\text{div } F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}.$$

To prove the second equality, we let $q = f_1$, $p = -f_2$ in Theorem 10.2.2. Then

$$\int\int_{\mathcal{B}} \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right) dA = \int_{\partial\mathcal{B}} -f_2 \, dx + f_1 \, dy.$$

Thus, if $\alpha(t) = (\alpha_1(t), \alpha_2(t))$, $a \leq t \leq b$, is a parametrization of $\partial\mathcal{B}$, we have

$$\int\int_{\mathcal{B}} \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right) da = \int_a^b (-f_2\alpha_1' + f_1\alpha_2') \, dt = \int_a^b F \cdot N \, ds.$$

Done. <<

This theorem has a nice physical interpretation. Think of F as the velocity vector of a fluid—a gas, say—moving in the plane, and assume unit

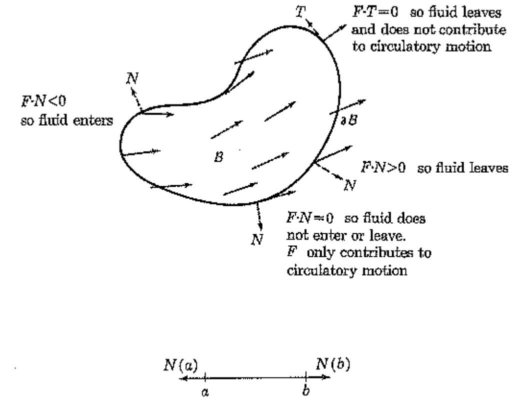

FIGURE 10.16. Stokes' Theorem in one dimension

density. Then div $F(X)$ measures how the gas is moving from the point X. Integrating this over the whole region $\mathcal{B}$, we obtain the net change in the amount of fluid in $\mathcal{B}$. But there is another way to measure the same quantity: merely stand by the walls of $\mathcal{B}$ and see how much leaves. See Figure 10.15. In computing this, we of course need only the component of the velocity F perpendicular to the boundary (the component of the velocity tangent to the boundary affects only the circulatory, that is, whirlpool-like, motion of the fluid in $\mathcal{B}$). This explains why we use the unit outer normal vector in the line integral. Theorem 10.2.4 is just a statement of the equality of these two ways of measuring the change in the amount of fluid in $\mathcal{B}$. If div $F = 0$, then the net fluid flow across any closed curve is zero. Thus, we refer to a vector field satisfying div $F = 0$ as having no sources or sinks in $\mathcal{B}$.

Perhaps a more familiar statement of the same idea is to let $\mathcal{B}$ denote a room in which a party is taking place and $F(X)$ the velocity of the person standing at X. Then one can measure the change in the number of people at the party either by (1) counting the people inside the room $\mathcal{B}$ or (2) standing at the doors ($= \partial\mathcal{B}$) and counting how many people enter and leave. The number from (1) is the left-hand side of Theorem 10.2.4, and (2) the right-hand side.

One can also see how the single-variable theorem

$$\int_a^b f'(x) \, dx = f(b) - f(a)$$

fits into this pattern. See Figure 10.16. The minus sign in front of $f(a)$ arises because the "outer normal" to the interval $a \leq x \leq b$ at a is in the negative direction.

We conclude this section with two important identities due to Green.

Green's first identity asserts that if $\mathcal{B}$ has a smooth boundary and if u and v are functions with continuous first and second partial derivatives, then

$$\iint_{\mathcal{B}} v\Delta u \, dA = \int_{\partial\mathcal{B}} v\nabla u \cdot N \, ds - \iint_{\mathcal{B}} \nabla v \cdot \nabla u \, dA,$$

where $\Delta u = u_{xx} + u_{yy}$ is called the **Laplacian** of u and $\nabla v = \text{grad } v = (v_x, v_y)$. This formula is an analog of integration by parts. To prove it, we use the Divergence Theorem 10.2.4 with $f_1 = vu_x$ and $f_2 = vu_y$. Then

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = vu_{xx} + v_x u_x + vu_{yy} + v_y u_y = v\Delta u + \nabla v \cdot \nabla u,$$

and the result follows. It is often useful to note that the term $\nabla u \cdot N$ in the boundary integral is the directional derivative of u in the direction of the outer normal to $\partial\mathcal{B}$. This is sometimes written $\partial u/\partial N$, and so Green's first identity reads

$$\iint_{\mathcal{B}} v\Delta u \, dA = \int_{\partial\mathcal{B}} v\frac{\partial u}{\partial N} \, ds - \iint_{\mathcal{B}} \nabla v \cdot \nabla u \, dA.$$

Green's second identity asserts that

$$\iint_{\mathcal{B}} (v\Delta u - u\Delta v) \, dA = \int_{\partial\mathcal{B}} \left(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N} \right) ds.$$

This is proved by writing Green's first identity with the roles of u and v reversed and then substracting the two equations.

10.2D Applications

Now let us see how Stoke's Theorem may be applied to some issues in mathematical physics. Let $\mathcal{B} \subseteq \mathbb{R}^2$ be a given region with smooth boundary, and let $u(x, y, t)$ denote the temperature at the point $(x, y) \in \mathcal{B}$ at time t. In physics, it is shown that a simple model for heat flow requires that u satisfy the differential equation $u_t = \Delta u$; that is,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

To determine the temperature $u(x, y, t)$ for $t \geq 0$, it is physically plausible to require prior knowledge of both the initial temperature (at $t = 0$)

$$u(x, y, 0) = f(x, y), \quad (x, y) \in \mathcal{B},$$

and the temperature on the boundary for all time $t \geq 0$

$$u(x, y, t) = \varphi(x, y, t), \quad (x, y) \in \partial\mathcal{B}, \quad t \geq 0.$$

Although we do not prove that a solution does exist, it is easy to show that there is at most one solution—a uniqueness theorem. Thus, once one solution satisfying the initial and boundary conditions has been found, by whatever means, you are guaranteed that there is no other one.

Suppose that v and w are two solutions, and let $u = v - w$. We show that $u(x, y, t) = 0$ for all $(x, y) \in \mathcal{B}$. This, of course, proves that $v = w$. Now, since $v_t = \Delta v$ and $w_t = \Delta w$, we see that

$$u_t - \Delta u = v_t - w_t - (\Delta v - \Delta w) = 0,$$

and for $(x, y) \in \mathcal{B}$,

$$u(x, y, 0) = v(x, y, 0) - w(x, y, 0) = f(x, y) - f(x, y) = 0,$$

and similarly for $(x, y) \in \partial\mathcal{B}$ and $t \geq 0$,

$$u(x, y, t) = v - w = \varphi - \varphi = 0.$$

Now—and here is the trick—we define a kind of "energy" function $E(t)$ by

$$E(t) = \frac{1}{2} \iint_{\mathcal{B}} u^2(x, y, t) \, dx \, dy.$$

Then, differentiating with respect to t under the integral sign (see the theorem in Section 9.3b), we find that

$$\frac{dE}{dt} = \frac{1}{2} \iint_{\mathcal{B}} \frac{\partial}{\partial t} u^2(x, y, t) dx \, dy = \iint_{\mathcal{B}} uu_t \, dx \, dy.$$

But $u_t = \Delta u$, and so by Green's first identity with $v = u$ there,

$$\frac{\partial E}{\partial t} = \iint_{\mathcal{B}} u\Delta u \, dx \, dy = \int_{\partial\mathcal{B}} u\nabla u \cdot N \, ds - \iint_{\mathcal{B}} \|\nabla u\|^2 \, dx \, dy.$$

Now we recall that $u = 0$ on $\partial\mathcal{B}$, and so the boundary integral is zero. Thus

$$\frac{dE}{dt} = -\iint_{\mathcal{B}} \|\nabla u\|^2 \, dx \, dy \leq 0.$$

Consequently, $E(t)$ is a decreasing function, $E(t) \leq E(0)$—physically, energy is dissipated. Since $u(x, y, 0) = 0$, however, we see that $E(0) = 0$. Moreover, it is evident from the definition that $E(t) \geq 0$. Hence for all $t \geq 0$

$$0 \leq E(t) \leq E(0) = 0;$$

that is, $E(t) = 0$ for all $t \geq 0$. This implies that $u(x, y, t) = 0$ for all $t \geq 0$, since if not, then $E(t) > 0$ for some $t \geq 0$.

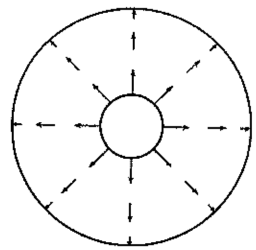

FIGURE 10.17. A region Ω with no sources or sinks

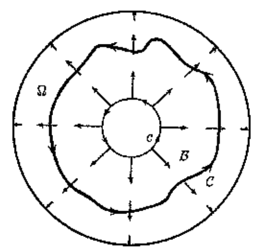

FIGURE 10.18. A closed curve in Ω

As our second application, we consider a vector field F satisfying div $F = 0$ in a region $\Omega \subseteq \mathbb{R}^2$; that is F has no sources or sinks in Ω. See Figure 10.17. For instance, we might have

$$F(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

with Ω the region $1 \le x^2+y^2 \le 9$ between two circles of radius 1 and 3, respectively. It is easy to check that, for $(x,y) \neq \mathbf{0}$,

$$\text{div } F = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = 0.$$

Let C be the curve in Figure 10.18. We claim that

$$\int_C F \cdot N \, ds = \int_c F \cdot N \, ds = 2\pi,$$

where c is the unit circle $\|X\| = 1$ oriented counterclockwise. If F represents the velocity vector of a fluid in Ω, this asserts that the net flow across the curve C equals that across the unit circle c. This is plausible if one observes that the vector field F is radial outward of length 1 at every point on c. It also appears that there is a source at the origin.

The computation is easy. Let $\mathcal{B}$ denote the region between C and the unit circle $\|X\| = 1$. Then by the Divergence Theorem 10.2.4,

$$0 = \iint_{\mathcal{B}} \text{div } F \, dA = \int_{\partial\mathcal{B}} F \cdot N \, ds.$$

But $\partial\mathcal{B} = C - c$, where we have $-c$, not c, since its orientation is reversed when considered part of the boundary of $\mathcal{B}$. Therefore

$$0 = \int_{\partial\mathcal{B}} F \cdot N \, ds = \int_C F \cdot N \, ds - \int_c F \cdot N \, ds.$$

We now evaluate the integral around c. There are two methods. The easiest is to observe that on c, the radius vector $X = (x,y)$ is the unit outer normal. Thus $F = N$ is also the unit outer normal, so that $F \cdot N = 1$ on c. Consequently

$$\int_C F \cdot N \, ds = \int_c ds = 2\pi.$$

The second method is to parametrize c as $x = \cos\theta,\ y = \sin\theta,\ 0 \le \theta \le 2\pi$. Then $F = (\cos\theta, \sin\theta)$ on c and, as we saw just before Theorem 10.2.4, $N = (\cos\theta, \sin\theta)$. Therefore $F \cdot N = 1$, and

$$\int_C F \cdot N \, ds = \int_0^{2\pi} 1 \, d\theta = 2\pi.$$

This number 2π represents the magnitude of the source at the origin.

Other applications of the various forms of Stokes' Theorem appear in later sections.

Exercises

1. As a check of Stokes' Theorem 10.2.2, compute both sides separately and showing that they are equal, for each of the following:
   (a) $F(x,y) = (1-y,\ x)$, $\mathcal{B}$ is the disk $x^2+y^2 \le 4$
   (b) $F(x,y) = (x+y,\ x-6y)$, $\mathcal{B}$ is the triangular region with vertices at $(-4,1)$, $(2,1)$, and $(2,5)$
   (c) $F(x,y) = (xy, -xy)$, $\mathcal{B}$ is the disk $x^2+y^2 \le 9$
   (d) $F(x,y) = (y\sin\pi x,\ y\cos\pi x)$, $\mathcal{B}$ is the rectangular region given by $1 \le x \le 2$ and $-1 \le y \le 1$
   (e) $F(x,y) = (x^2+y,\ e^y - x)$, $\mathcal{B}$ is the region above the parabola $y = x^2$ and below the line $y = 1$
   (f) $F(x,y) = (x^2+y^2,\ -2xy)$, $\mathcal{B}$ is the half-disk $x^2+y^2 \le 16,\ y \ge 0$
   (g) $F(x,y) = (x,\ 2x+y)$, $\mathcal{B}$ is the ring region $1 \le x^2+y^2 \le 4$
   (h) $F(x,y) = (y^2,\ x^2)$, $\mathcal{B}$ is the region outside the circle $x^2+y^2 = 1$ but inside the rectangle $-2 \le x \le 3,\ -4 \le y \le 5$.

2. (a) Given a bounded region $\mathcal{B} \subseteq \mathbb{R}^2$, show that

$$\text{Area } (\mathcal{B}) = \frac{1}{2} \int_{\partial \mathcal{B}} x\, dy \; - \; y\, dx \, .$$

(b) Use part (a) to find the area inside the ellipse

$$(x, y) = (a \cos\varphi, b \sin\varphi)\, , \;\; 0 \le \varphi \le 2\pi \, .$$

(c) Use part (a) to find the area of the region $1 \le x^2 + y^2 \le 25$.

3. If $u : \mathbb{R}^2 \to \mathbb{R}$ is a function with continuous first and second partial derivatives, prove that

$$\int\!\!\int_{\mathcal{B}} \Delta u \; dA = \int_{\partial \mathcal{B}} \frac{\partial u}{\partial N} \; ds.$$

4. Let a force field $F = (2 - 5x + y, x)$. Find the work expended in moving a particle of unit mass counterclockwise around the boundary of the triangle with vertices at $(1, 1)$, $(3, 1)$, and $(3, 5)$.

5. Let C_1 denote the circle $x^2 + y^2 = 1$ and C_2 the circle $(x - 2)^2 + (y - 1)^2 = 25$, both oriented counterclockwise, and let $\mathcal{B}$ denote the ring region between these curves. If a vector field F satisfies div $F = 0$ in $\mathcal{B}$, show that

$$\int_{C_1} F \cdot N \; ds = \int_{C_2} F \cdot N \; ds.$$

(Note that the proof of this extends immediately to the case where C_1 and C_2 are replaced by more general curves and $\mathcal{B}$ is the region between them.)

6. For certain kinds of heat flow, the temperature $u(x, y, t)$ in a region $\mathcal{B} \subset \mathbb{R}^2$ satisfies

$$u_t = \Delta u - u.$$

Prove that there is at most one solution that has a given initial temperature $u(x, y, 0) = f(x, y)$, $(x, y) \in \mathcal{B}$, and a given boundary temperature $u(x, y, t) = \varphi(x, y, t)$ for $(x, y) \in \partial \mathcal{B}$ and $t \ge 0$. (The same function $E(t)$ given in Section 10.2d works here.)

7. Let $F = (p, q)$ be a vector field with a continuous derivative in $\mathbb{R}^2$, and let $\mathcal{B} \subseteq \mathbb{R}^2$ be a region with a smooth boundary $\partial \mathcal{B}$. If F is a constant vector on $\partial \mathcal{B}$, show that

$$\int\!\!\int_{\mathcal{B}} (q_x - p_y) \; dA = 0 \, .$$

10.3 Surface integrals

10.3A Smooth Oriented Surfaces in $\mathbb{R}^3$

Stokes' Theorem generalizes to three and higher dimensions. We content ourselves with an intuitive description of the $\mathbb{R}^3$ case without proofs. For this case we require the notion of integration on an oriented smooth surface Σ in $\mathbb{R}^3$.

Let Σ be a surface in $\mathbb{R}^3$. We picture Σ as consisting of two disjoint pieces, the **surface interior**, which we will denote by Σ_o, and the **surface boundary**, which we denote by $\partial\Sigma$. Suppose that the surface boundary $\partial\Sigma$ is the union of a finite number of closed curves, and further suppose that at each point X on the surface interior Σ_o, there is a tangent plane. Then we call Σ a **smooth surface**. If the surface boundary of Σ is empty, then we call Σ a **closed surface**.

Examples:

1. The torus and the sphere in $\mathbb{R}^3$ are examples of surfaces Σ such that $\partial\Sigma = \emptyset$, since both of these surfaces have tangent planes at every point. Thus they are both closed surfaces.

2. The upper hemisphere

$$\Sigma = \{(x, y, z) : x^2 + y^2 + z^2 = 1 \text{ and } z \ge 0\}$$

has the set

$$\Sigma_o = \{(x, y, z) : x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}$$

as its surface interior, and the circle

$$\partial\Sigma = \{(x, y, z) : x^2 + y^2 + z^2 = 1 \text{ and } z = 0\}$$

as its surface boundary.

3. Consider the set

$$\Sigma = \{(x, y, z) : z^2 = x^2 + y^2 \text{ and } a \ge z \ge b\}\, ,$$

where $a > b > 0$. This is the circular cone $z = \|(x, y)\|$ truncated by the planes $z = a$ and $z = b$. It looks somewhat like an upside-down lampshade. Its surface interior is

$$\Sigma_o = \{(x, y, z) : z^2 = x^2 + y^2 \text{ and } a > z > b\}\, ,$$

and its surface boundary consists of two horizontal circles centered on the z-axis. One of these circles has radius a and lies in the plane $z = a$, and the other has radius b and lies in the plane $z = b$.

4. If we let $b = 0$ in the preceding example, we obtain a surface that looks like an ice cream cone (or a dunce cap). One piece of the surface boundary is the same as before, namely a circle of radius a in the plane $z = a$. The other piece has shrunk to a single point, namely the origin. You may not find it natural to think of this point as a closed curve, or even as a surface boundary point. However, it is the image of the function $F(t) = (0, 0, 0)$ (with $0 \le t \le 1$, say), so it is a curve. And for the purposes of this discussion, we want to consider it a surface boundary point, since there is no tangent plane to the cone at the origin.

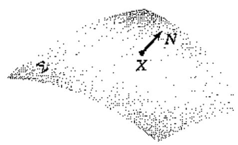

FIGURE 10.19. A surface in $\mathbb{R}^3$

To orient a smooth surface Σ, we must designate one side of the surface as the positive side. We do this by supposing that at each point X on the surface interior Σ_o there is attached a unit vector $N = N(X)$ that is perpendicular to the tangent plane at X. See Figure 10.19. The direction of N is called the **orientation** of Σ at X, and N is called the **positive unit normal vector** of Σ at X. The adjective "positive" signifies that N points in the direction of the orientation of Σ. The vector $-N$ is the **negative** unit normal of Σ at X. Note that N is a vector-valued function from Σ_o to $\mathbb{R}^3$. If N is a *continuous* function, we say that Σ is **oriented** by N.

EXAMPLE:

5. Not all surfaces can be oriented. The "Moebius strip" is a famous example of this. The Moebius strip may be parametrized as follows:

$$G(\theta,t) = ((1+t\cos\theta)\cos\theta, (1+t\cos\theta)\sin\theta, t\sin\theta)\,,$$

where $0 \le \theta \le 2\pi$ and $-\frac{1}{2} \le t \le \frac{1}{2}$. Ask someone to show you a few of the strange and fascinating properties of this surface. We guarantee that he or she will be delighted to do so.

Once a surface Σ is oriented, we are automatically led to an orientation of the surface boundary $\partial\Sigma$ by thinking of Σ as a piece of a "curved plane". See Figure 10.20, where Σ looks like part of a curved plane with a hole cut out of it. Imagine that you are standing on the positive side of an oriented surface, with your right foot on one of the curves that makes up its surface boundary, and your left side toward the surface interior. Then you are facing in the direction of the orientation of the curve. Try to visualize yourself doing this near each of the two boundary curves in Figure 10.20. Note that if you stand on the opposite (negative) side of Σ near one of the curves and face in the direction of orientation of the curve, then the surface interior will be on your *right*, and your left foot will be on the boundary curve.

We now describe a natural way in which a surface can be oriented by using a parametrization. Suppose $G = (g_1(s,t), g_2(s,t), g_3(s,t))$ is a parametrization of the surface Σ in $\mathbb{R}^3$. Let (s_0,t_0) be a point in the domain of G where G is differentiable. We recall the notation

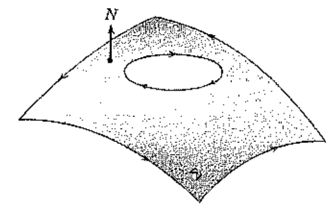

FIGURE 10.20. An oriented surface

$$D_1G(s,t) = \frac{\partial G}{\partial s}(s,t) = \left(\frac{\partial g_1}{\partial s}(s,t)\,, \frac{\partial g_2}{\partial s}(s,t)\,, \frac{\partial g_3}{\partial s}(s,t)\right)$$

and

$$D_2G(s,t) = \frac{\partial G}{\partial t}(s,t) = \left(\frac{\partial g_1}{\partial t}(s,t)\,, \frac{\partial g_2}{\partial t}(s,t)\,, \frac{\partial g_3}{\partial t}(s,t)\right)$$

from Sections 8.3 and 9.5. If the linear transformation $G'(s_0,t_0)$ has rank 2, or in other words, if the vectors $D_1G(s_0,t_0)$ and $D_2G(s_0,t_0)$ are linearly independent, Theorem 8.3.2 tells us that the plane $\mathcal{P}$ parametrized by

$$sD_1G(s_0,t_0) + tD_2G(s_0,t_0) + G(s_0,t_0)$$

is tangent to the image of G at the point (s_0,t_0). Let N be the unit vector that has the same direction as the cross product $D_1G(s_0,t_0) \times D_2G(s_0,t_0)$. Then N is orthogonal to the two vectors $D_1G(s_0,t_0)$ and $D_2G(s_0,t_0)$, and since those two vectors have directions parallel to the tangent plane $\mathcal{P}$, it follows that N points in a direction that is perpendicular to the tangent plane of Σ at $X = G(s_0,t_0)$. Thus the parametrization G leads to an orientation of the surface at the point $G(s_0,t_0)$.

Now suppose for each point (s,t) of the domain of G such that $G(s,t)$ is on the surface interior Σ_o, the derivative $G'(s,t)$ is continuous and has rank 2. For $X = G(s,t)$, let

$$N(X) = \frac{D_1G(s,t) \times D_2G(s,t)}{\|D_1G(s,t) \times D_2G(s,t)\|}\,.$$

Since $G'(s,t)$ is continuous, the vector-valued function N is continuous, *provided* it is well-defined. A problem with the definition may arise if G maps more than one point in its domain to the same point X on the surface, for then it would be possible to get two different normal vectors $N(X)$. If we obtain the same vector $N(X)$ from the above formula for all points in the domain of G that are mapped to X, then $N(X)$ is well-defined on Σ_o, and we say that Σ is oriented **by the parametrization** G.

EXAMPLES:

6. The upper half of the unit sphere $S(\mathbf{0};1)$ in $\mathbb{R}^3$ is oriented by the parametrization

$$G(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)\,,$$

where $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi/2$. Let us determine which side of the sphere is the positive side under this orientation. We compute the positive normal vector at the point $(\sqrt{2}/2, 0, \sqrt{2}/2) = G(0,\pi/4)$. The vectors $D_1G(0,\pi/4)$ and $D_2G(0,\pi/4)$ are computed by taking partial derivatives with respect to θ and φ. We get

$$D_1G(0,\pi/4) = (0,\sqrt{2}/2,0) \text{ and } D_2G(0,\pi/4) = (\sqrt{2}/2,0,-\sqrt{2}/2)\,,$$

so $N(\sqrt{2}/2,0,\sqrt{2}/2)$ is the unit vector in the direction of the vector

$$(0,\sqrt{2}/2,0)\times(\sqrt{2}/2,0,-\sqrt{2}/2) = (-1/2,0,-1/2)\,.$$

The third coordinate of this vector is negative, so it points toward the "inside" of the hemisphere. Similar computations can be made at other points, and in each case, the positive normal vector has a negative third coordinate. Thus, the inside of the hemisphere is its positive side under this particular orientation. The orientation of the surface boundary, the "equator", is clockwise around the vertical axis (when viewed from above). Note that if we parametrize the equator by setting $\varphi = \pi/2$,

$$F(\theta) = G(\theta,\pi/2)\,,$$

then F gives us the reverse of this orientation. (This is a coincidence, not a general rule.)

7. You will be asked to show in the exercises that the parametrization given earlier for the Moebius band *does not* produce a well-defined unit normal vector at every point. It is in fact true that if a smooth surface Σ has a parametrization G that with a continuous derivative of rank 2 at every point in its domain, and if G produces two different normal vectors $N(X)$ at some point X on Σ_o, then Σ cannot be oriented.

REMARKS:

- If a surface Σ is oriented by a parametrization $G(s,t)$, then it is oriented in the opposite direction by the parametrization $H(s,t) = G(t,s)$, since the positive normal vector for the parametrization H points in the direction of

$$\begin{aligned} D_1H(s,t)\times D_2H(s,t) &= D_2G(t,s)\times D_1G(t,s) \\ &= -D_1G(s,t)\times D_2G(s,t)\,. \end{aligned}$$

In other words, simply by switching the order of the variables, we change the orientation to the opposite direction. Try this with the parametrization of the hemisphere given above.

- In the example of the hemisphere, we glossed over a minor technical point, namely that the vectors D_1G and D_2G are not linearly independent when $\varphi = 0$, since $D_1G(\theta,0) = \mathbf{0}$ for all θ. When $\varphi = 0$, $G(\theta,\varphi)$ is the "north pole" $(0,0,1)$, so G fails to provide us with a unit normal vector at $(0,0,1)$. There are several ways to take care of this problem, but perhaps it is best to either not worry about it, or to think of such a point as a surface boundary point, such as we did for the vertex of the cone in Example 4. Note that there is in fact a tangent plane at the north pole, and that the orientation provided by G can be extended continuously to give the unit normal vector $N(0,0,1) = (0,0,-1)$ at that point. A similar technicality occurs at the south pole when we extend this parametrization to the whole sphere by letting φ range between 0 and π.

10.3B DEFINITION OF A SURFACE INTEGRAL

We are now ready to discuss the integral of a vector field F in $\mathbb{R}^3$ over an oriented surface Σ in $\mathbb{R}^3$. We assume that Σ is a smooth oriented surface, with positive unit normal vector $N(X)$ at each point X in Σ_o. Imagine that Σ is partitioned into tiny pieces Σ_i that have surface area ΔA_i. Let X_i be a point on Σ_i. Then the sum

$$\sum_i F(X_i)\cdot N(X_i)\Delta A_i$$

is to be our approximation of the **surface integral**

$$\int_\Sigma F(X)\cdot N(X)\,dA\,.$$

You should think about the analogy between this description of the surface integral and our development of the line integral. A completely rigorous definition of the surface integral is beyond the scope of this book.

REMARK:

- One physical interpretation of the quantity $F(X)\cdot N(X)\,dA$ is in terms of wind pressure on a surface, such as a sail. The vector field F indicates the direction and strength of the wind at each point. The inner product $F(X)\cdot N(X)$ gives the force per unit area exerted on a surface perpendicular to $N(X)$, and the quantity $F(X)\cdot N(X)\,dA$ is the force exerted on the small piece of the sail at the point X with surface area dA. Note that if the force $F(X)$ is perpendicular to $N(X)$, then the inner product is 0, and no force is exerted on the surface at X. This is as expected.

Just as with line integrals, in order to compute a surface integral, we need a parametrization. Thus we assume that Σ is oriented by a parametrization G. To derive a formula for surface integrals on Σ, we think of each of the small pieces Σ_i as images under G of rectangles in the domain of G. Let $\mathcal{R}$ be the domain of G, and partition $\mathcal{R}$ (at least approximately) into small rectangles $\mathcal{R}_i$ centered at (s_i, t_i), with sides of length Δs_i and Δt_i parallel to the coordinate axes in $\mathbb{R}^2$. Let $\Sigma_i = G(\mathcal{R}_i)$. Then Σ_i has approximately the same area as the parallelogram $G'(s_i, t_i)(\mathcal{R}_i)$ in $\mathbb{R}^3$. According to Theorem 3.7.5, the area of the parallelogram is

$$\|D_1G(s_i,t_i) \times D_2G(s_i,t_i)\| \cdot (\text{ area of } \mathcal{R}_i)\ .$$

Letting $X_i = (s_i, t_i)$, it follows that

$$\sum_i F(X_i) \cdot N(X_i)\Delta A_i \approx$$
$$\sum_i F(G(s_i,t_i)) \cdot N(G(s_i,t_i))\|D_1G(s_i,t_i) \times D_2G(s_i,t_i)\|\Delta s_i \Delta t_i\ .$$

Using our formula for N in terms of the parametrization G, we have

$$\sum_i F(X_i) \cdot N(X_i)\Delta A_i \approx \sum_i F(G(s_i,t_i)) \cdot (D_1G(s_i,t_i) \times D_2G(s_i,t_i))\ .$$

This reasoning justifies the following theorem:

Theorem 10.3.1 Let F be a continuous vector field defined on $\mathbb{R}^3$, and let Σ be a smooth surface in $\mathbb{R}^3$ that is oriented by a parametrization G with domain $\mathcal{R}$. Then

$$\begin{aligned} \int_\Sigma F(X) \cdot N(X)\,dA &= \int\int_{\mathcal{R}} F(G(s,t)) \cdot (D_1G(s,t) \times D_2G(s,t))\,dA \\ &= \int\int_{\mathcal{R}} [F \circ G\,,\ D_1G\,,\ D_2G]\,dA\ . \end{aligned}$$

(Recall that $[X, Y, Z]$ is the notation for the triple product of three vectors in $\mathbb{R}^3$.)

EXAMPLE:

8. We compute the surface integral of $F(x,y,z) = (y,x,z)$ over the portion of the cone $z = \|(x,y)\|$ where $z \leq 3$. We orient the cone so that the positive side is the "outside". Thus all the positive unit normal vectors will have *negative* third coordinate. A parametrization for this orientation is

$$G(\theta, t) = (t\cos\theta, t\sin\theta, t)\ ,$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq t \leq 3$. At the point $G(\theta, t)$, we have

$$\begin{aligned} D_1G(\theta,t) \times D_2G(\theta,t) &= (-t\sin\theta, t\cos\theta, 0) \times (\cos\theta, \sin\theta, 1) \\ &= (t\cos\theta, t\sin\theta, -t)\ . \end{aligned}$$

Note that this vector has negative third coordinate. Now we compute the integrand

$$F(G(\theta,t)) \cdot D_1G(\theta,t) \times D_2G(\theta,t) = t^2(\sin 2\theta - 1)\ .$$

This integrand is easily integrated over the rectangle determined by $0 \leq \theta \leq 2\pi$, $0 \leq t \leq 3$ in the θt-plane. The answer is -18π.

10.3C Stokes' Theorem in $\mathbb{R}^3$

In this section we give two versions of Stokes' Theorem for surface integrals. For the first version, we need a definition. If $F = (p, q, r)$ is a differentiable vector field in $\mathbb{R}^3$, we define curl F to be the vector field

$$\text{curl } F = (r_y - q_z, p_z - r_x, q_x - p_y)\ .$$

EXAMPLE:

9. Since the curl F is a new object, we compute it for $F(X) = (xyz,\ y - 3z,\ 2y)$. Then $p = xyz$, $q = y - 3z$, and $r = 2y$, so that curl $F = (2+3,\ xy - 0,\ 0 - xz) = (5,\ xy,\ -xz)$.

We remark that the complicated formula for curl F is usually remembered by thinking of it as the symbolic cross product of the operator $\nabla = (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j} + (\partial/\partial z)\mathbf{k}$ with the vector F and is written

$$\text{curl } F = \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ p & q & r \end{vmatrix}\ .$$

Theorem 10.3.2 (Stokes' or Ostrogradsky's Theorem) Let

$$F(x,y,z) = (p(x,y,z), q(x,y,z), r(x,y,z))$$

be a differentiable vector field in $\mathbb{R}^3$. Then for any smooth oriented surface Σ

$$\int\int_\Sigma (\text{curl } F) \cdot N\ dA = \int_{\partial\Sigma} F(X) \cdot dX\ .$$

REMARK:

- In the special case when Σ is a region in the xy-plane, so that $N = (0,0,1)$, $r = 0$, and $F = (p,q,0)$, then
$$(\text{curl } F) \cdot N = q_x - p_y,$$
and the formula simplifies to
$$\iint_\Sigma (q_x - p_y)\, dA = \int_{\partial\Sigma} F(X) \cdot dX,$$
which is just our Theorem 10.2.2.

The second generalization of Stokes' Theorem is the three-dimensional version of the Divergence Theorem. Let $F = (p,q,r)$ be a differentiable vector field in $\mathbb{R}^3$. Then, by definition,
$$\text{div } F = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z}$$
Let $\mathcal{B} \subset \mathbb{R}^3$ be a solid region whose "exterior" $\partial\mathcal{B}$ is an oriented surface with positive unit normal N pointing "away from" $\mathcal{B}$, then the Divergence Theorem states that
$$\iiint_\mathcal{B} \text{div } F\, dV = \iint_{\partial\mathcal{B}} F \cdot N\, dA,$$
where $dV = dx\, dy\, dz$ is the element of volume in $\mathbb{R}^3$. The physical interpretation made before in terms of fluid flow carries over to this three-dimensional case without change.

It is straightforward to prove the three-dimensional versions of Green's identities by imitating the two-dimensional proofs. For example, Green's first identity becomes
$$\iiint_\mathcal{B} v\Delta u\, dV = \iint_{\partial\mathcal{B}} v\nabla u \cdot N\, dA - \iiint_\mathcal{B} \nabla v \cdot \nabla u\, dV,$$
where $\Delta u = u_{xx} + u_{yy} + u_{zz}$ and $\nabla v = (v_x, v_y, v_z)$.

Although it might be difficult to believe, these theorems are quite easy to use in most standard applications. Here are a few.

EXAMPLES:

10. Let $F = (x, 3y, -z)$, and let Σ be the sphere $S((0,0,0);a)$. Then, if $\mathcal{B} = B((0,0,0);a)$ is the solid ball of radius a, we have $\Sigma = \partial\mathcal{B}$, and
$$\iint_\Sigma F \cdot N\, dA = \iiint_\mathcal{B} \text{div } F\, dV = 4\pi a^3.$$
To see this, we compute div $F = 1 + 3 - 1 = 3$, and so the right side is three times the volume of the ball; that is, $3(4\pi a^3/3) = 4\pi a^3$ as claimed.

11. Let $u(x,y,z)$ be a function that satisfies the **Laplace equation**
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0;$$
that is, $\Delta u = 0$, in a domain $\mathcal{B}$ in $\mathbb{R}^3$. Such functions are called **harmonic functions**. They arise often in applications. For example, u might be the temperature distribution in a body at thermal equilibrium (this means that the temperature is independent of time). If the temperature is zero on the boundary of the body, $u = 0$ on $\partial\mathcal{B}$, we claim that it must be zero throughout the body. This physically plausible assertion is easy to prove. We apply Green's first identity, stated above, in the special case when $u = v$. Since $\Delta u = 0$, the integral on the left is zero. Moreover, $u = 0$ on $\partial\mathcal{B}$. Thus the double integral over $\partial\mathcal{B}$ is also zero. This leaves
$$\iiint_\mathcal{B} \|\nabla u\|^2\, dV = 0,$$
which implies that $\nabla u = 0$. That is, $u' = O$ throughout $\mathcal{B}$. Therefore $u \equiv$ constant in $\mathcal{B}$. But $u = 0$ on $\partial\mathcal{B}$, and so the value of the constant is zero; that is, $u \equiv 0$ throughout $\mathcal{B}$. The proof is complete.

10.3D GRAVITATIONAL FORCE

The purpose of this section is to briefly investigate the gravitational force due to a spherically symmetric solid. We will apply some of the results obtained so far to prove that in this case the gravitational force is the same as that due to a point mass. Although the problem caused Newton considerable difficulty, we find it quite easy. To use Newton's phrase, however, "We are standing on the shoulders of giants".

Recall (Section 5.3) Newton's law of gravitation for the force F on a mass m at X due to a mass M at Y:
$$F = -\gamma \frac{mM}{\|X - Y\|^3}(X - Y).$$
It is convenient to introduce the force at X per unit mass, $G = F/m$,
$$G(X) = -\gamma \frac{M}{\|X - Y\|^3}(X - Y).$$
$G(X)$ is called the **gravitational field** at X. By a straightforward computation, one sees that
$$\text{div } G(X) = 0, \quad \text{for } X \neq Y.$$

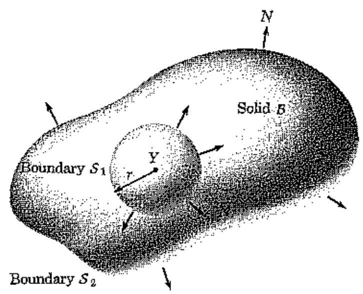

FIGURE 10.21. A point mass contained in a hole in $\mathcal{B}$

Thus if $\mathcal{B} \subseteq \mathbb{R}^3$ is any region whose boundary is a finite union of closed smooth surfaces, and if $\mathcal{B}$ does not contain Y (see Figure 10.21, where Y is in a hole in $\mathcal{B}$), then the Divergence Theorem yields

$$0 = \iiint_{\mathcal{B}} \operatorname{div} G\, dV = \iint_{\partial\mathcal{B}} G \cdot N\, dA = \iint_{\mathcal{S}_2} G \cdot N\, dA - \iint_{\mathcal{S}_1} G \cdot N\, dA.$$

Here we have used $-\mathcal{S}_1$, since the orientation given to $\mathcal{S}_1$ in Figure 10.21 (its positive normal points into $\mathcal{B}$) is the negative normal when regarded as part of the boundary of $\mathcal{B}$ (denoted $\partial\mathcal{B}$ above). Consequently, for the smooth surfaces $\mathcal{S}_1, \mathcal{S}_2$, with $Y \notin \mathcal{B}$,

$$\iint_{\mathcal{S}_1} G \cdot N\, dA = \iint_{\mathcal{S}_2} G \cdot N\, dA.$$

Now we pick a convenient surface for $\mathcal{S}_1$: a sphere of radius r with center at Y. Then for $X \in \mathcal{S}_1$, the positive unit normal to $\mathcal{S}_1$ is $N = (X - Y)/(\|X - Y\|)$ and $r = \|X - Y\|$. Therefore, if $X \in \mathcal{S}_1$,

$$G \cdot N = -\frac{-\gamma M}{\|X - Y\|^2} = -\frac{\gamma M}{r^2}.$$

Now, since the area of $\mathcal{S}_1 = 4\pi r^2$, we have

$$\iint_{\mathcal{S}_2} G \cdot N\, dA = \iint_{\mathcal{S}_1} G \cdot N\, dA = -\frac{\gamma M}{r^2} \iint_{\mathcal{S}_1} dA = -4\pi\gamma M.$$

All this has assumed that the mass M (at Y) lies inside the surface $\mathcal{S}_1$. On the other hand, if M lies outside $\mathcal{S}_2$, then $\operatorname{div} G = 0$ throughout $\mathcal{B}$. See Figure 10.22. Hence

$$\iiint_{\mathcal{B}} \operatorname{div} G\, dV = \iint_{\mathcal{S}_2} G \cdot N\, dA = 0.$$

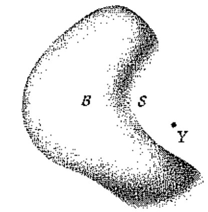

FIGURE 10.22. A point mass at Y outside the surface $\mathcal{S}$

Combining these two cases, we conclude that if $\mathcal{S}$ is any closed surface, then

$$\iint_{\mathcal{S}} G \cdot N\, dA = \begin{cases} 0, & M \text{ outside } \mathcal{S}, \\ -4\pi\gamma M, & M \text{ outside } \mathcal{S}. \end{cases}$$

It is important to observe that this integral, called the total "flux" of the gravitational field through the surface $\mathcal{S}$, depends only on the mass M and whether M is inside or outside $\mathcal{S}$, but not on the particular location Y of the mass nor on the shape of the surface.

Now suppose that there are lots of masses $M_1, M_2, \ldots, M_n$ at the points $Y_1, Y_2, \ldots, Y_n$. Let $G_j(X)$ be the gravitational field at X due to the mass M_j at Y_j. Since force is a vector, the gravitational field G due to all these masses is the sum

$$G = G_1 + \ldots + G_n.$$

Consequently, if $\mathcal{S}$ is any closed surface (see Figure 10.23), then the result above applied to each mass separately yields

$$\begin{aligned} \iint_{\mathcal{S}} G \cdot N\, dA &= \sum_1^n \iint_{\mathcal{S}} G_j \cdot N\, dA \\ &= -4\pi\gamma \cdot \text{ (total mass inside } \mathcal{S}); \end{aligned}$$

that is,

$$\iint_{\mathcal{S}} G \cdot N\, dA = -4\pi\gamma \cdot \text{ (total mass inside } \mathcal{S}).$$

This is called **Gauss's Law**.

More generally yet, if the mass distribution in a region $\mathcal{B}$ is given in terms of a density ρ, then

$$M = \text{ total mass in } \mathcal{B} = \textstyle\iiint_{\mathcal{B}} \rho\, dV.$$

Gauss's Law clearly extends to this case, as one can see by thinking of M as the sum $\sum \Delta M$, where $\Delta M = \rho\, \Delta V$ is the mass in a small element of volume ΔV.

We now apply this to the case of a spherical body $\mathcal{B}$, like the earth, and see how to compute the gravitational field. We assume that the mass

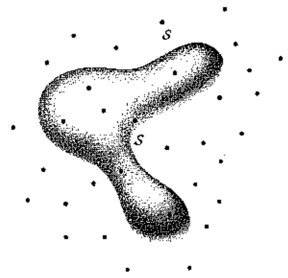

FIGURE 10.23. Many point masses outside the surface $\mathcal{S}$

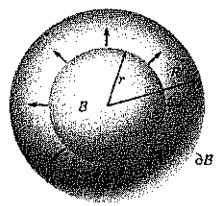

FIGURE 10.24. Spherically symmetric mass distribution

distribution is symmetric and place the origin at the center of $\mathcal{B}$. See Figure 10.24. This implies, by symmetry, that the gravitational force $F(X)$ and hence the gravitational field $G(X)$ are in the radial direction. Consequently, if $\mathcal{S}_r$ is any sphere of radius r concentric with $\partial\mathcal{B}$, possibly inside $\mathcal{B}$, we have $G \cdot N = \|G\|$, where N is the unit outer normal, in the radial direction, to $\mathcal{S}_r$. Moreover, again by symmetry, we see that $\|G(X)\|$ is constant for all X on $\mathcal{S}_r$. Therefore

$$\int\!\!\int_{\mathcal{S}_r} G \cdot N \, dA = \|G\| \int\!\!\int_{\mathcal{S}_r} dA = 4\pi r^2 \|G\|.$$

Combined with Gauss's Law, we find that

$$\|G(X)\| = -\frac{\gamma M_r}{r^2} = -\frac{\gamma M_r}{\|X\|^2},$$

for X on $\mathcal{S}_r$. Here M_r is the total mass inside the sphere $\mathcal{S}_r$. Since the direction of $G(X)$ is radial, that is, in the direction of the unit vector $X/\|X\|$, we finally conclude that

$$G(X) = -\frac{\gamma M_r}{\|X\|^3} X,$$

and so

$$F(X) = -\frac{\gamma m M_r}{\|X\|^3} X.$$

Of course, if $r \geq R$, where R is the radius of $\mathcal{B}$, then $M_r = M$, the total mass of $\mathcal{B}$. Since, given any point X, we can construct the sphere $\mathcal{S}_r$ with $r = \|X\|$, the last formula proves that the gravitational force at X due to a spherically symmetric mass distribution is the same as that from a point mass M_r at the center of the body, where M_r is the mass of the body at distance $\leq r = \|X\|$ from the center of the body.

Exercises

1. Let $F(x, y, z) = (2xy, yz, z)$ be a vector field in $\mathbb{R}^3$. Use the divergence form of Stokes' Theorem to evaluate
$$\int\!\!\int_{\Sigma} F \cdot N \, dA,$$
where Σ is the boundary of the cubical region $\|x\| \leq 1,\ \|y\| \leq 1,\ \|z\| \leq 1$.
2. Let $F(x, y, z) = (x - y,\ y^2 - z,\ x^2 + 5z)$ be a vector field in $\mathbb{R}^3$. Evaluate
$$\int\!\!\int_{\Sigma} F \cdot N \, dA,$$
where Σ is the sphere $x^2 + y^2 + z^2 = 9$.
3. Compute curl F for the following vector fields F:
   (a) $F(x, y, z) = (y,\ z,\ 2 - z)$
   (b) $F(x, y, z) = (xy + z,\ yz + x,\ zx + y)$
   (c) $F(x, y, z) = (x^2 + y^2,\ 0,\ 0)$
   (d) $F(x, y, z) = (y - z,\ z + x,\ x + y)$.
4. Let $F = (p, q, r)$ be a differentiable vector field in $\mathbb{R}^3$ and u a real-valued function. Prove the following:
   (a) div (curl F) $= 0$
   (b) curl (grad u) $= 0$
   (c) curl $(uF) = u\,$curl $F +$ grad $u) \times F$
   (d) if U is a constant vector, then curl $(U \times (x, y, z)) = 2U$.
5. Let F and G be differentiable vector fields in $\mathbb{R}^3$. Show that:
   (a) div $(F \times G) =$ (curl $F) \cdot G - F \cdot$ curl G
   (b) curl $(F + G) =$ curl $F +$ curl G
   (c) if curl $F = G$, then div $G = 0$.
6. Is there a vector field $F = (p, q, r)$ such that
   (a) curl $F = (x, y, z)$? Why?
   (b) curl $F = 2(z - y,\ x - z,\ y - x)$? Why?
7. Indicate on a sketch the orientation given to the torus by the parametrization in Example 2 of Section 8.0.

8. Find a parametrization of $S((0,0,0);1)$ that gives it an outward orientation.

9. Let
$$\Sigma = \{(x,y,z) : x^2+y^2 = 1 \text{ and } -1 \le z \le 1\}.$$

   (a) Draw Σ.
   (b) Find a parametrization of Σ that gives it an "outward" orientation. Indicate this orientation on your sketch of Σ, along with the corresponding orientation of the surface boundary curves.
   (c) Calculate the surface integral of $F(x,y,z) = (x,y,z)$ over Σ. Give an interpretation of this result. *Hint:* What is the surface area of Σ?
   (d) Calculate the surface integral of $F(x,y,z) = (x^2+y^2, 0, xyz)$ over Σ.
   (e) Show that the third coordinate function of F in the preceding part is irrelevant. Can you explain why?
   (f) Use your answer to part (d) to determine the value of the surface integral of the constant function $F(x,y,z) = (1,0,0)$ over Σ without further integration. Can you give an interpretation of this number in terms of the surface area of a simple geometric object?
   (g) Compute the surface integral of
$$F(x,y,z) = (5, xy, -xz) = \text{ curl } (xyz, y-3z, 2y)$$
   over Σ, first by using the general formula for surface integrals, and second by using Stokes' Theorem.

10. Calculate a unit normal vector at each point of the surface interior of the Moebius strip of Example 5. Verify that the parametrization given there produces two different normal vectors at some points on the surface interior.

11. Let g be a scalar-valued function defined on a simple region $\mathcal{R}$ in $\mathbb{R}^2$, and let Σ be the graph of g. Assume that ∇g exists, is not equal to $\mathbf{0}$, and is continuous at all points in the interior of the domain of g.

   (a) Show that Σ is oriented by the unit normal vector
$$N(x,y,z) = \frac{(-D_1g(x,y), -D_2g(x,y), 1)}{\|(-D_1g(x,y), -D_2g(x,y), 1)\|}.$$
   This orients Σ so that its "top" is the positive side.
   (b) Use the answer to the previous part to derive a formula for the surface integral of a vector field F over the surface Σ.

12. Calculate the surface integrals of the vector field F over the oriented surface Σ.

   (a) $F(x,y,z) = (e^{x+y+z}, xy, z+1)$, Σ oriented by the parametrization $G(s,t) = s(0,1,-2) + t(1,1,3) + (2,-1,0)$, where (s,t) lies in the triangle bounded by the lines $s=0, t=0, s+t=1$
   (b) $F(x,y,z) = (yz, x+z, 2)$, Σ is the graph of $g(x,y) = x^2 - y^2$, $0 \le x, y \le 1$, oriented upward

   (c) $F(X) = X$, Σ is the torus of Example 2 in Section 8.0, oriented outward.

13. Let Σ be the hemisphere $x^2+y^2+z^2 = 1$, and $z \ge 0$, oriented outward, and let $F = (x-y, 3z^2, -x)$. Evaluate
$$\iint_\Sigma (\text{curl } F) \cdot N \, dA.$$

14. Let Σ be the part of the plane $x+y+z=1$ in the first octant of $\mathbb{R}^3$, oriented so that the positive unit normal is $N = (1,1,1)/\sqrt{3}$. As a check on Stokes' Theorem, evaluate both sides of
$$\iint_\Sigma (\text{curl } F) \cdot N \, dA = \int_{\partial\Sigma} F(X) \cdot dX,$$
where $F = (z-y, x-z, y-x)$.

15. Let $\mathcal{B} \subseteq \mathbb{R}^3$ be a solid region, say a ball, and F a vector field with a continuous derivative. Show that
$$\iint_{\partial\mathcal{B}} (\text{curl } F) \cdot N \, dA = 0.$$
Hint: Let $\Sigma = \partial\mathcal{B}$, and observe that $\partial\Sigma$ has no points.

16. Let $\mathcal{B} \subseteq \mathbb{R}^3$ denote the solid pyramid bounded by the planes $x+y+2z = 6$, $x=0$, $y=0$, and $z=0$. Use the Divergence Theorem to evaluate
$$\iint_{\partial\mathcal{B}} F \cdot N \, dA,$$
where $F = (2x, 2y, 4z)$.

17. Let F and G be two vector fields with continuous derivatives in a ball $\mathcal{B} \subseteq \mathbb{R}^3$ such that (1) $F = \nabla\varphi$, $G = \nabla\psi$, and (2) $\text{div } F = \text{div } G$ in $\mathcal{B}$, while (3) $F \cdot N = G \cdot N$ on the boundary of $\mathcal{B}$, where N is the positive unit normal to $\partial\mathcal{B}$. Prove that $F \equiv G$ in $\mathcal{B}$.

18. (a) Let $\Sigma \subseteq \mathbb{R}^3$ be the sphere $S((0,0,0);1)$ and Σ' be some smooth closed surface outside the sphere Σ, so that for instance Σ' might some larger sphere. If $\text{div } F = 0$ in the region between Σ and Σ', show that
$$\iint_\Sigma F \cdot N \, dA = \iint_{\Sigma'} F \cdot N \, dA.$$
   (b) Let $\mathcal{B} \subseteq \mathbb{R}^3$ be the region $1 \le \|X\| \le 3$. If $F(X) = X/\|X\|^3$ in $\mathcal{B}$, show by direct computation of both sides that
$$\iiint_\mathcal{B} \text{div } F \, dV = \iint_{\partial\mathcal{B}} F \cdot N \, dA.$$

19. Prove the Divergence Theorem in $\mathbb{R}^3$ if $\mathcal{B}$ is a box: $a_1 \le x \le b_1$, $a_2 \le y \le b_2$, $a_3 \le z \le b_3$.

20. Let $\mathcal{B}$ be a hollow spherical shell. Show that the gravitational force inside $\mathcal{B}$ is zero.

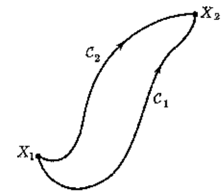

FIGURE 10.25. Two different paths from X_1 to X_2

10.4 Independence of Path; Potential Functions

10.4A The Issue

Say that X_1 and X_2 are points in $\mathbb{R}^q$ and F is a force field on $\mathbb{R}^q$. Then the work done by the force F in moving a unit mass along a curve $\mathcal{C}$ from one endpoint X_1 to the other endpoint X_2, is

$$\text{Work} = \int_{\mathcal{C}} F(X) \cdot V(X)\, ds.$$

The question is, under what conditions on F is the work independent of the particular oriented curve $\mathcal{C}$ from X_1 to X_2? See Figure 10.25. In physical problems, it is customary to refer to an oriented curve as a **path**. We then rephrase the question: under what conditions of F is the work independent of the path? In Example 8 in Section 10.1c, the line integral *does* depend on the path, and so it would be wrong to believe that one has independence of the path unless further restrictions are imposed on the force F. If the line integral is independent of the path $\mathcal{C}$, we say that F is a **conservative force field**. The reason for the name is easy to explain. If F is conservative, then the line integrals along any two paths $\mathcal{C}_1$ and $\mathcal{C}_2$ from X_1 to X_2 are equal. Therefore, if we go from X_1 to X_2 along $\mathcal{C}_1$ and return from X_2 to X_1 along $-\mathcal{C}_2$, the net result is zero; that is, no work is needed to traverse the closed curve $\mathcal{C}_1 - \mathcal{C}_2$. Consequently, the force F could not have any dissipative aspects, like friction, since dissipation causes an irreversible loss of energy.

The question of a line integral being independent of the path can be raised about any line integral. It does not need the physical conception of work.

10.4B The Main Theorems

We would like some criterion to determine when a line integral is independent of the path. First we show that some line integrals are indeed independent of the path.

Theorem 10.4.1 Let $F = \nabla\varphi$, where φ is a scalar-valued function defined on $\mathcal{D}(F) \in \mathbb{R}^n$, and let X_1 and X_2 be any two points in $\mathcal{D}(F)$. Then

$$\int_{X_1}^{X_2} F \cdot V\, ds = \int_{X_1}^{X_2} \nabla\varphi \cdot V\, ds = \varphi(X_2) - \varphi(X_1)\,,$$

where the integration on the left-hand side is over any piecewise smooth path $\mathcal{C}$ from X_1 to X_2 that lies in $\mathcal{D}(F)$. In particular the integral on the left-hand side depends on the path only through its endpoints, provided that F is a gradient.

Proof: First assume that $\mathcal{C}$ is smooth, and let $X = \alpha(t)$, $a \le t \le b$, be a parametrization of $\mathcal{C}$. Note that $\alpha(a) = X_1$, $\alpha(b) = X_2$. Then

$$\int_{\mathcal{C}} F \cdot V\, ds = \int_a^b \nabla\varphi(\alpha(t)) \cdot \alpha'(t)\, dt\,.$$

But by the Chain Rule

$$\frac{d}{dt}\varphi(\alpha(t)) = \nabla\varphi(\alpha(t)) \cdot \alpha'(t)$$

Therefore, by the Fundamental Theorem of Calculus,

$$\int_{\mathcal{C}} F \cdot V\, ds = \int_a^b \frac{d}{dt}\varphi(\alpha(t))\, dt = \varphi(X_2) - \varphi(X_1)\,.$$

Since the right-hand side is independent of the path $\mathcal{C}$ from X_1 to X_2, we conclude that the line integral is independent $\mathcal{C}$..

If $\mathcal{C}$ is only piecewise smooth, then let its smooth segments be

$$X_1Z_1,\ Z_1Z_2, \ldots, Z_nX_2\,.$$

See Figure 10.26. We apply the result above to each smooth segment.

$$\int_{X_1}^{X_2} \nabla\varphi \cdot V\, ds = \int_{X_1}^{Z_1} + \int_{Z_1}^{Z_2} + \cdots + \int_{Z_n}^{X_2}$$
$$= [\varphi(Z_1) - \varphi(X_1)] + [\varphi(Z_2) - \varphi(Z_1)] + \cdots + [\varphi(X_2) - \varphi(Z_n)]\,.$$

After cancellation, we find that

$$\int_{X_1}^{X_2} \nabla\varphi \cdot V\, ds = \varphi(X_2) - \varphi(X_1)\,,$$

just as desired. This completes the proof. <<

We remark that this theorem is itself a generalization of the Fundamental Theorem of Calculus, to which it reduces if $\mathcal{C}$ is an interval on the x_1-axis

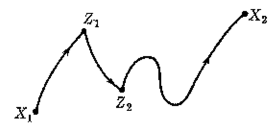

FIGURE 10.26.

and $f(X)$ depends on x_1 only. The next result is our main theorem. In it we collect a variety of conditions for the line integral of a given vector field F to be independent of the path of integration. It asserts that essentially the only way this can happen is if there is some function φ such that $F = \nabla\varphi$, in which case Theorem 10.4.1 gives the independence of path. Recall that a curve $\mathcal{C}$, parametrized by $X = \alpha(t)$, $a \leq t \leq b$, is closed if the initial and final points coincide: $\alpha(a) = \alpha(b)$.

Theorem 10.4.2 Let $F = (f_1, \ldots, f_n)$ be a vector field in a pathwise connected open subset $\mathcal{B}$ in $\mathbb{R}^n$. Assume that each of the coordinate functions of F has continuous first partial derivatives in $\mathcal{B}$. The following conditions are equivalent:

1. $\int_{\mathcal{C}} F \cdot V\, ds = 0$ for any piecewise smooth closed curve in $\mathcal{B}$.
2. $\int_{\mathcal{C}} F \cdot V\, ds$ is independent of the path for any piecewise smooth curve $\mathcal{C}$ joining two fixed points in $\mathcal{B}$.
3. There is a function φ such that $\nabla\varphi = F$;

Any of these implies the following condition on F:

4. $D_j f_i = D_i f_j$ for $1 \leq i, j \leq n$.

Moreover, if $\mathcal{B}$ is an open ball, then condition (4) implies conditions (1), (2), and (3). Thus, for an open ball $\mathcal{B}$, conditions (1) through (4) are equivalent.

PROOF: We prove the chain of implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). This will prove the equivalence of (1), (2), and (3).

(1) $\Rightarrow$ (2). Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be any two piecewise smooth curves from X_1 to X_2. Then $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$ is a piecewise smooth closed curve in $\mathcal{B}$, which we have pictured as having a "hole." See Figure 10.27. Therefore, by (1),

$$0 = \int_{\mathcal{C}} F \cdot V\, ds = \int_{\mathcal{C}_1} + \int_{-\mathcal{C}_2} = \int_{\mathcal{C}_1} - \int_{\mathcal{C}_2},$$

So that

$$\int_{\mathcal{C}_1} F \cdot V\, ds = \int_{\mathcal{C}_2} F \cdot V\, ds .$$

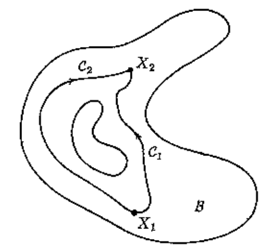

FIGURE 10.27. A piecewise smooth closed curve in $\mathcal{B}$

(2) $\Rightarrow$ (3). We must exhibit the function $\varphi(X)$. Pick a point X_0 in $\mathcal{B}$ and fix it throughout the discussion. If X is in $\mathcal{B}$, let

$$\varphi(X) = \int_{X_0}^{X} F(Y) \cdot dY ,$$

where we need not specify the path of integration since, by part (2), the result does not depend on the particular path chosen. Note that there is at least one path from X_0 to X since $\mathcal{B}$ is pathwise connected. It can be shown that since $\mathcal{B}$ is open, such a path can be chosen to be smooth.

Fix a point $X_1 \in \mathcal{B}$. We must show that φ is differentiable at X_1 and that $\nabla\varphi(X_1) = F(X_1)$, that is, $T(X) = \varphi(X_1) + F(X_1) \cdot (X - X_1)$ is the best affine approximation to φ at X_1. Now

$$\varphi(X) - T(X) = \int_{X_0}^{X} F(Y) \cdot dY - \int_{X_0}^{X_1} F(Y) \cdot dY - F(X_1) \cdot (X - X_1)$$

so

$$\frac{\varphi(X) - T(X)}{\|X - X_1\|} = \frac{1}{\|X - X_1\|} \int_{X_1}^{X} [F(Y) - F(X_1)] \cdot dY .$$

Since F is differentiable, and hence continuous, one can pick a ball $B(X_1; \delta)$ so small that $\|F(Y) - F(X_1)\| < \varepsilon$ for $Y \in B(X_1; \delta)$. By Theorem 10.1.3,

$$\left| \int_{X}^{X_1} [F(Y) - F(X_1)] \cdot dY \right| \leq \|X - X_1\| \varepsilon ,$$

so

$$\frac{|\varphi(X) - T(X)|}{\|X - X_1\|} \leq \varepsilon$$

for all $X \in \mathcal{B}$.

(3) $\Rightarrow$ (1). This follows from Theorem 10.4.1, since for a closed curve $\mathcal{C}$ in that theorem, we have $X_2 = X_1$.

To prove part (4), assume that (3) holds. That is, assume that $F = \nabla\varphi$. We observe that, by part (3), $D_i\varphi = f_i$, and $D_j\varphi = f_j$. By assumption, F

has continuous first partial derivatives, so φ has continuous second partial derivatives. Therefore, because of the theorem about the equality of mixed partial derivatives (Theorem 6.3.1 in Section 6.3c), we have

$$D_j f_i = D_{ij}\varphi = D_{ji}\varphi = D_i f_j .$$

Finally, let us assume that $\mathcal{B}$ is an open ball, and prove in this case that (4) $\Rightarrow$ (3). Without loss of generality, we assume that the ball $\mathcal{B}$ is centered at the origin. For $X \in \mathcal{B}$, define

$$\begin{aligned}\varphi(X) &= \int_0^1 F(tX)\cdot X\,dt \\ &= \int_0^1 [x_1 f_1(tX) + \ldots + x_n f_n(tX)]\,dt .\end{aligned}$$

We are using the fact that $\mathcal{B}$ is a ball to ensure that the point tX is in $\mathcal{B}$ for all $t \in [0,1]$.

We wish to compute the partial derivatives of φ. To accomplish this, we use the theorem in Section 9.3 about differentiating under the integral sign. We have

$$\frac{\partial\varphi}{\partial x_i}(X) = \int_0^1 \frac{\partial}{\partial x_i}[x_1 f_1(tX) + \cdots + x_n f_n(tX)]\,dt .$$

Using the chain rule to differente term by term in the integrand on the right-hand side, we obtain

$$\frac{\partial\varphi}{\partial x_i}(X) = \int_0^1 [f_i(tX) + t(x_1 D_i f_1(tX) + \ldots + x_n D_i f_n(tX))]\,dt .$$

Now we use the hypothesis that $D_i f_j = D_j f_i$. The integrand becomes

$$f_i(tX) + t(x_1 D_1 f_i(tX) + \ldots + x_n D_n f_i(tX)) ,$$

and according to Example 4 in Section 8.2, this equals

$$\psi(X,t) + t\frac{\partial\psi(X,t)}{\partial t} = \frac{d[t\psi(X,t)]}{dt} ,$$

where $\psi(X,t) = f_i(tX)$. Substitute this expression into the integral and use the Fundamental Theorem of Calculus to obtain

$$\frac{\partial\varphi}{\partial x_i}(X) = t f_i(tX)\Big|_{t=0}^{t=1} = f_i(X) .$$

It follows that $\nabla\varphi(X) = F(X)$ as desired. Done. <<

REMARK:

- If the line integral of a force field F is independent of the path, we say that the force is conservative and call the function φ such that $\nabla\varphi = F$ the **potential function** of F.

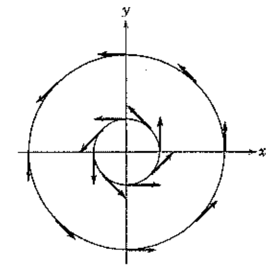

FIGURE 10.28. A ring in $\mathbb{R}^2$

10.4C EXAMPLES

One wonders if the proof that (4) $\Rightarrow$ (3) can be extended to more general regions than balls. The following example shows that it cannot always be generalized.

EXAMPLE:

1. Let $\mathcal{B} \subseteq \mathbb{R}^2$ be the ring domain $1 \le x^2+y^2 \le 9$, and let $F(x,y) = (p(x,y), q(x,y))$, where

$$p(x,y) = -\frac{y}{x^2+y^2}, \qquad q(x,y) = \frac{x}{x^2+y^2} .$$

Then a computation shows that $p_y = q_x$. See Figure 10.28. If we integrate $F = (p,q)$ around the circle $x = 2\cos\theta$, $y = 2\sin\theta$, $0 \le \theta \le 2\pi$, then

$$\int_C F\cdot V\,ds = \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\,d\theta = 2\pi .$$

In other words, the line integral of F around closed curve C is not zero. Thus (4) does not imply (1) in this case.

It turns out that (4) implies (1) only if the region $\mathcal{B}$ has no "holes" in it. In the example just done, $\mathcal{B}$ does have a hole, namely the disk $x^2+y^2 < 1$, and at the center of this hole, the given vector field F "blows up". This is typical of the problems that arise if the region has one or more holes.

EXAMPLES:

2. Let C be a smooth curve from $X_1 = (2,3)$ to $X_2 = (3,-1)$. We evaluate

$$\int_C (3x^2+y)\,dx + (e^y + x)\,dy .$$

There are three methods for evaluating this. All three rely on the observation that the integral is independent of the path, since $p_y = 1 = q_x$ for all $(x, y) \in \mathbb{R}^2$ (here $\mathcal{B}$ is a big disk, or even all of $\mathbb{R}^2$).

Method 1. We pick a convenient path—the straight line C:

$$x = 2 + t, \quad y = 3 - 4t, \quad 0 \le t \le 1.$$

Then

$$\begin{aligned} \int_C &= \int_0^1 [3(2+t)^2 + (3-4t) - 4e^{3-4t} - 4(2+t)]\, dt \\ &= 10 + e^{-1} - e^3 . \end{aligned}$$

Method 2. Since $p_y = q_x$ and $\mathcal{B}$ is a disk containing X_1 and X_2, by Theorem 10.4.2, there is a function φ such that $\nabla\varphi = F$,

$$\nabla\varphi = (\varphi_x, \varphi_y) = (3x^2 + y,\ e^y + x) .$$

We find φ by the method used in the proof of Theorem 10.4.2. Thus

$$\begin{aligned} \varphi(x,y) &= \int_0^1 (3t^2x^2 + ty\, ,\, e^{ty} + tx) \cdot (x,y)\, dt \\ &= (x^3 + (xy/2)) + (e^y - 1 + (xy/2)) \\ &= x^3 + xy - 1 + e^y . \end{aligned}$$

The line integral we are calculating equals

$$\begin{aligned} &\varphi(3,-1) - \varphi(2,3) \\ &= (27 - 3 - 1 + e^{-1}) - (8 + 6 - 1 + e^3) = 10 + e^{-1} - e^3 . \end{aligned}$$

Method 3. The beginning and end of this method is similar to the beginning and end of the preceding method. The difference is in the way that the potential function φ is found. Since $\varphi_x = 3x^2 + y$, by integrating with respect to x and holding y constant, we find that

$$\varphi(x,y) = \int (3x^2 + y)\, dx = x^3 + xy + K(y) ,$$

where $K(y)$ is a function (the "constant" of integration) that depends only on y. Now we compute φ_y from this expression and compare it with what φ_y is known to be, that is, $\varphi_y = e^y + x$:

$$e^y + x = \varphi_y = x + \frac{dK}{dy}.$$

This yields $K'(y) = e^y$, and so $K(y) = e^y$, where we have ignored the integration constant since we are only looking for *one* potential function, that is,

$$\varphi(x,y) = x^3 + xy + e^y.$$

Even though the potential function obtained here differs by a constant from the one found by the preceding method, they give identical answers for the line integral.

3. Let $F(x_1, x_2, x_3) = (2x_2 + 2x_3\, ,\ 2x_3 + 2x_1\, ,\ 2x_1 + 2x_2)$. We show that F is a conservative vector field and find its potential function. Since $D_i f_j = D_j f_i$ for each i and j, we conclude from Theorem 10.4.2 that F is conservative. We use the method given in the proof of Theorem 10.4.2 to calculate a potential function φ. Thus

$$\begin{aligned} &\varphi(x_1,x_2,x_3) \\ &= \int_0^1 (2tx_2 + 2tx_3\, ,\ 2tx_3 + 2tx_1\, ,\ 2tx_1 + 2tx_2) \cdot (x_1,x_2,x_3)\, dt \\ &= (x_2 + x_3)x_1 + (x_3 + x_1)x_2 + (x_1 + x_2)x_3 \\ &= 2(x_1x_2 + x_2x_3 + x_3x_2) . \end{aligned}$$

You might wonder why one prefers to work with potential functions instead of vector fields. One answer is that potential functions are scalars and it is easier to work with scalar-valued functions, like $2(x_1x_2 + x_2x_3 + x_3x_1)$, than with vector-valued functions.

Exercises

1. Evaluate the line integrals $\int_C F \cdot V\, ds$ by finding and using a potential function:
   (a) $F(x,y) = (x,y)$, C is the image of $(\cos t, 2\sin t)$ for $0 \le t \le \pi/2$
   (b) $F(X) = (x_3\, ,\ x_2{+}x_3\, ,\ x_1{+}x_2{+}3)$, C is the image of $(t^2, t, t{+}1)$ for $-1 \le t \le 2$
   (c) $F(x,y) = (e^y,\ xe^y)$, C is the boundary of the rectangular region given by $-1 \le x \le 2$ and $5 \le y \le 9$, beginning at $(2,5)$ and oriented counterclockwise
   (d) $F(x,y) = (e^y,\ xe^y)$, C is the boundary of the rectangular region given by $-1 \le x \le 2$ and $5 \le y \le 9$, beginning at $(-1,9)$ and oriented counterclockwise
   (e) $F(x_1,x_2) = (x_2^2,\ 2x_1x_2{+}2x_2)$, C is the semicircle defined by $x_1^2{+}x_2^2 = 9$ and $x_1 \ge 0$, oriented clockwise
   (f) $F(x,y) = (x+y, x-7y)$, C is a pentagon with successive vertices at $(2,0)$, $(1,1)$, $(0,1)$, $(-1,0)$, and $(0,-1)$, oriented clockwise.
2. (a) Show that the force field $F(X) = (x{+}e^x \sin y,\ e^x \cos y)$ is conservative and find a potential function for it.
   (b) Let C be a smooth curve from $(1,\pi)$ to $(0,\pi/2)$. Evaluate the work performed by the force to move a particle of unit mass along C. Recall that

$$\text{Work} = \int_C F \cdot V\, ds.$$

3. Let C be a smooth path in the disk $x^2 + (y-3)^2 \leq 6$ joining the two points $(-2, 2)$ and $(2, 2)$. For which of the following vector fields does the line integral depend on C only through its endpoints?

   (a) $F(X) = (x - 1,\ e^y)$

   (b) $F(X) = (3x - y,\ 2y + x)$

   (c) $F(X) = \left(-\frac{y}{x^2+y^2},\ \frac{x}{x^2+y^2}\right)$

   (d) $F(X) = (e^y \cos x,\ e^y \sin x)$

   (e) $F(X) = (e^y, e^x)$

   (f) $F(X) = \left(\frac{x}{1+x^2+y^2},\ \frac{y}{1+x^2+y^2}\right)$.

4. Let $F(X) = \left(-\frac{y}{x^2+y^2},\ \frac{x}{x^2+y^2}\right)$. Evaluate $\int_C F \cdot V\, ds$, where:

   (a) C is the shorter arc of the circle $x^2 + y^2 = 8$ from $(-2, 2)$ to $(2, 2)$.

   (b) C is the longer arc of the circle $x^2 + y^2 = 8$ from $(-2, 2)$ to $(2, 2)$.

   (c) Is the line integral independent of the path C in the disk $x^2 + y^2 \leq 100$ from $(-2, 2)$ to $(2, 2)$. Does this agree with the "moreover" part of Theorem 10.4.2?

5. The gravitational force between a mass M at the origin in $\mathbb{R}^3$ and a mass m at X in $\mathbb{R}^3$ is
$$F(X) = \frac{-\gamma m M X}{\|X\|^3},$$
where γ is a constant. Show that F is conservative by finding a potential function φ) such that $F = \text{grad}\ \varphi$ at all points other than $\mathbf{0}$. See Exercise 16 in Section 10.1.

6. Let C be a smooth closed curve, and let p and q be functions having continuous first derivatives on $\mathbb{R}$. Show that
$$\int_C p(x)\ dx + q(y)\ dy = 0.$$

7. Show that $\int_C (x-y)\ dx + 2y\ dy$ is not independent of the path C by showing that condition (4) in Theorem 10.4.2 does not hold.

8. Decide whether the following statement is true or false and explain your answer, including a counterexample in case your answer is "false". If F is a conservative vector field and G is any continuous vector field, then for any smooth closed curve C
$$\int_C (F(X) + G(X)) \cdot V(X)\, ds = \int_C G(X) \cdot V(X)\, ds.$$

9. Decide whether the following statement is true or false and explain your answer, including a counterexample in case your answer is "false". Let F and G be continuous vector fields in $\mathbb{R}^2$. If F is conservative and if the vectors $F(X)$ and $G(X)$ have the same direction at every point $X \in \mathbb{R}^2$, then G is conservative too.

10. Let ψ be a scalar-valued function with a continuous derivative on the interval $(0, \infty)$. Let $F(X) = \psi(\|X\|)X$ for $X \in \mathbb{R}^n$ different from $\mathbf{0}$. Prove that F is conservative by verifying that $F(X) = \nabla\varphi(\|X\|)$, where
$$\varphi(r) = \int_0^r \rho\varphi(\rho)\ d\rho\,.$$
In particular, since any central force field F is of the form above, this shows that *central force fields are conservative*. Compare with Exercise 18 of Section 10.1.