Linear Maps from $\mathbb{R}^2$ to $\mathbb{R}^3$

As an exercise, which I hope you will (soon) realize is entirely routine, we will show that a linear map $F(X) = Y$ from $\mathbb{R}^2$ to $\mathbb{R}^3$ must just be three linear high school equations in two variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= y_1 \\ a_{21}x_1 + a_{22}x_2 &= y_2 \\ a_{31}x_1 + a_{32}x_2 &= y_3 \end{aligned} \tag{1}$$

Linearity means for any vectors U and V in R^2 and any scalars c

$$F(U+V) = F(U) + F(V) \qquad \text{and} \qquad F(cU) = cF(U).$$

Idea: write $X := (x_2,\, x_2) \in \mathbb{R}^2$ as

$$X = x_1(1,0) + x_2(0,1) = x_1e_1 + x_2e_2, \qquad \text{where} \qquad e_1 := (1,0), \quad e_2 := (0,1)$$

(physicists often write e_1 as $\mathbf{i}$ and e_2 as $\mathbf{j}$ but using this notation in higher dimensions one quickly runs out of letters).
Then, by the two linearity properties

$$\begin{aligned} Y = F(X) &= F(x_1e_1 + x_2e_2) \\ &= F(x_1e_1) + F(x_2e_2) \\ &= x_1F(e_1) + x_2F(e_2). \end{aligned}$$

But $F(e_1)$ and $F(e_2)$ are just specific vectors in $\mathbb{R}^3$ so this last equation is exactly the desired (1) with

$$F(e_1) = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \qquad \text{and} \qquad F(e_2) = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}.$$

Collecting the ingredients we have found that

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = Y = F(x) = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = \begin{pmatrix} x_1a_{11} + x_2a_{12} \\ x_1a_{21} + x_2a_{22} \\ x_1a_{31} + x_2a_{32} \end{pmatrix}$$

as claimed in(1).

[Last revised: September 5, 2012]