Midterm Exam 1

Math 312, Fall 2012

You have 50 minutes.

No books, phones, calculators, talking, etc. You are allowed your own hand-written 8.5×11 inch note sheet, but **no other materials**. You may write on the back side of the sheets if necessary.

PLEASE WRITE LEGIBLY AND SHOW YOUR WORK.
YOU MAY NOT RECEIVE FULL CREDIT IF YOU DO NOT SHOW WORK.

Good luck!

Name ________________________________

Total Score ____________ (/XX points)

1. *(XX points)* Suppose V and W are vector spaces. What exactly does it mean for a function $T : V \to W$ to be linear?

2. *(XX points)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that sends $\vec{e}_1$ to $\vec{e}_1 + \vec{e}_3$, $\vec{e}_2$ to $-\vec{e}_1$, and finally $\vec{e}_3$ to to $-\vec{e}_2 + \vec{e}_3$.

   (a) Find the matrix representation of T (using the standard basis).

   (b) Describe what the inverse transformation T^{-1} does to each of the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$. (This will involve some computations.)

   (c) Find all solutions $\vec{x}$ to the equation $T(\vec{x}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

3. *(XX points)* Let R, S, and T be linear transformations from $\mathbb{R}^2 \to \mathbb{R}^2$ that perform the following operations:

- R rotates vectors by π radians counter-clockwise.
- S reflects across the line $y = x$.
- T preserves $\vec{e}_1$ but skews $\vec{e}_2$ to $\vec{e}_1 + \vec{e}_2$.

Find a matrix representation for the linear transformation that first applies T, then S, then finally R.

4. *(XX points)* Let A be a 5×3 matrix, so the function $T(\vec{x}) = A\vec{x}$ is a linear transformation. Answer the following questions **with a brief explanation**.

(a) Is $A\vec{x} = \vec{b}$ necessarily solvable for all $\vec{b}$ in $\mathbb{R}^5$?

(b) Suppose the kernel of T is one-dimensional. How many linearly independent columns does A have?

(c) Could it be possible to find a 3×5 matrix B such that AB is the identity matrix?

5. *(XX points)* Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation that sends a polynomial $p(x)$ to $p''(x) - 2p(x)$

(a) Find the matrix representation $[T]_{\mathcal{B}}$ of T using the basis $\mathcal{B} = \{x^2, x, 1\}$.

(b) Find a basis for the kernel of T, using your matrix $[T]_{\mathcal{B}}$.

(c) Find a basis for the image of T, using your matrix $[T]_{\mathcal{B}}$.

(d) Is T an isomorphism? Why or why not?

6. *(XX points)* Which of the following sets are vector spaces? **Explain**. For those sets that are vector spaces **write down a basis**.

   (a) The set of 2×2 matrices whose trace equals 1. (Recall the trace of a matrix is the sum of its diagonal entries.)

   (b) The set of polynomials $p(x)$ of degree at most 2 satisfying $p'(1) = 0$.

   (c) In $\mathbb{R}^2$, the span of the linearly dependent vectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

   (d) The set of 2×3 matrices whose rows each sum to zero.

7. *(XX points)* Let S, T be linear transformations from a vector space V to itself. Suppose each of S and T have a 1-dimensional kernel and that S and T commute (meaning $S \circ T = T \circ S$).

   (a) What are the possible dimensions of the kernel of $T \circ S$? Explain.

   (b) Let $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be differentiation on the vector space of infinitely differentiable functions, and let I be the identity transformation. Find a basis for the kernel of $(D - I)$. (Hint: if you apply $(D - I)$ to a function f, you obtain $f' - f$.)

   (c) Find a basis for the kernel of $(D + 2I)$.

   (d) Find a basis for the kernel of $(D - I) \circ (D + 2I)$.