Signature Printed Name

My signature above certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

Math 312 | Dec. 6, 2012

Exam 3

Jerry L. Kazdan | 12:00 – 1:20

Directions This exam has two parts, Part A, shorter problems, has 5 problem (6 points each so 30 points). Part B has 6 standard problems (10 points each, so 60 points). Total is 90 points. Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. Please justify your answers with clear reasons. No credit will be given to "correct" answers with either no or incorrect reasons.

Part A: Short Problems (5 problem, 6 points each).

1. Give an example of a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ with the property that $A^4 = I$ but $A^2 \neq I$.

   Solution: Perhaps the simplest example is a matrix that rotates a vector by 90 degrees: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

   The following is *almost* an example: $M := \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, where $i = \sqrt{-1}$. However this matrix has complex numbers so M maps real vectors to complex vectors, which is not what was sought.

2. Let V and W be linear spaces and $A : V \to W$ a linear map. Show that the image of A is a linear space.

   Solution: Let $\vec{w}_1$ and $\vec{w}_2$ be in the image of A. We need to show that $c\vec{w}_1$ and $\vec{w}_1 + \vec{w}_2$ are in the image of A for any scalar c.

   Since $\vec{w}_1$ and $\vec{w}_2$ are in the image of A, there are vectors $\vec{v}_1$ and $\vec{v}_2$ in V such that $A\vec{v}_1 = \vec{w}_1$ and $A\vec{v}_2 = \vec{w}_2$. Therefore

   $$A(c\vec{v}_1) = cA\vec{v}_1 = c\vec{w}_1 \qquad \text{and} \qquad A(\vec{v}_1 + \vec{v}_2) = \vec{w}_1 + \vec{w}_2$$

   so both $c\vec{w}_1$ and $\vec{w}_1 + \vec{w}_2$ are in the image of A.

3. Let A be a square matrix. If A^2 is invertible, must A be invertible? Proof or counterexample.

   Solution: **Method 1:** Since A^2 is invertible then $\det A^2 \neq 0$. But $\det A^2 = (\det A)(\det A)$ so $\det A \neq 0$. Therefore A is invertible.

   Method 2: Since A is a square matrix, it is invertible if and only if $ker(A) = 0$. But if $A\vec{v} = 0$ for some $\vec{v} \neq 0$, then $A^2\vec{v} = 0$ which contradicts the invertibility of A^2.

   Method 3: Since A is a square matrix, it is invertible if and only if it is onto, that is, for any $\vec{y}$ there is a solution of $A\vec{x} = \vec{y}$. But there is a solution of $A^2\vec{u} = \vec{y}$. But then $A(A\vec{u}) = \vec{y}$. Thus $\vec{x} = A\vec{u}$ is the desired solution.

Method 4: This is perhaps the best method since it explicitly *exhibits* the desired inverse. Some students in the class devised it – but I had not.

We know that $(A^2)^{-1}$ exists and has the property that

$$I = (A^2)^{-1}A^2 = [(A^2)^{-1}A]A.$$

Thus the expression in brackets, $B := (A^2)^{-1}A$ is the desired A^{-1} because $I = BA$. Note that also $AB = I$, either by beginning with $I = A^2(A^2)^{-1}$ or else by using the fact that for a *square matrix*, a left inverse, $I = BA$ (which is equivalent to A being one-to-one), is also a right inverse, $I = AB$ (which is equivalent to A being onto).

REMARK: The following is *not* a proof: $(A^2)^{-1} = (A^{-1})^2$ since this proof relies on the identity $(AB)^{-1} = B^{-1}A^{-1}$ which requires the prior assumption that both A and B are invertible. The formula $(AB)^{-1} = B^{-1}A^{-1}$ is valuable to show that if both A and B are invertible, then so is AB, but it does not imply the converse.

The following is a standard example where $AB = I$ but neither A nor B are invertible:

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is true that if A and B are both $n \times n$ matrices (so they are both *square* matrices of the same size), then $AB = I$ implies that both A and B are invertible. This follows from either of Methods 1, 2, or 3 above.

4. Let A be a matrix all of whose eigenvalues are 1. If A is diagonalizable, show that A must be the identity matrix, $A = I$.

   SOLUTION: Since A is diagonalizable, then $S^{-1}AS = D$ for some invertible matrix S, where D is a diagonal matrix with the eigenvalues of A on the diagonal. Thus $A = SDS^{-1}$. In this case, $D - I$. Thus $A = SIS^{-1} = I$.

   There is an equally simple proof using the diagonalizable of A as meaning there is a basis of eigenvectors of A.

   REMARK: The matrix $B := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ has all its eigenvalue are 1 yet it is not the identity matrix. It is not diagonalizable.

5. Let A be real matrix with a real eigenvalue λ and corresponding eigenvector $\vec{v}$. Similarly let μ be an eigenvalue of A^* with corresponding eigenvector $\vec{w}$. If $\mu \neq \lambda$, show that $\vec{v}$ and $\vec{w}$ are orthogonal.

   SOLUTION:

   $$\langle A\vec{v},\, \vec{w}\rangle = \langle \lambda\vec{v},\, \vec{w}\rangle = \lambda\langle \vec{v},\, \vec{w}\rangle \qquad \text{and} \qquad \langle \vec{v},\, A^*\vec{w}\rangle = \langle \vec{v},\, \mu\vec{w}\rangle = \mu\langle \vec{v},\, \vec{w}\rangle.$$

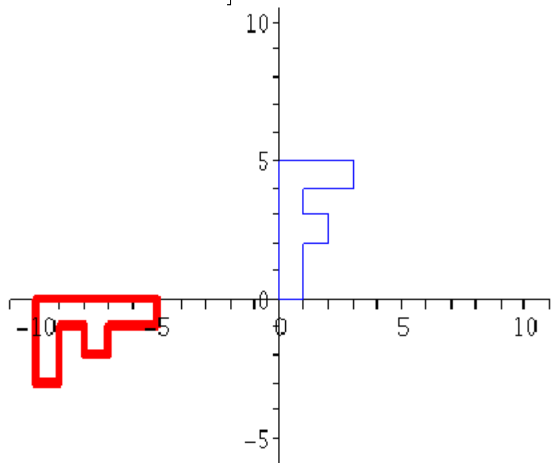
   But $\langle A\vec{v},\, \vec{w}\rangle = \langle \vec{v},\, A^*\vec{w}\rangle$. Therefore

   $$\lambda\langle \vec{v},\, \vec{w}\rangle = \mu\langle \vec{v},\, \vec{w}\rangle.$$

   Since $\lambda \neq \mu$, then $\langle \vec{v},\, \vec{w}\rangle = 0$.

Part B: Traditional Problems (6 problems, 10 points each so 60 points)

B–1. For the following figure find a matrix A and vector V that gives the indicated transformation $TX = V + AX$. [The new **F** is bold.]



SOLUTION: Since $T0 = V$ and from the figure $T : \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} -5 \\ 0 \end{pmatrix}$, we find that $V = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$.

To find A, we note that $AX = TX - V$ is the same map, only keeping the origin fixed. There are several ways to find A.

Method 1 Observe that $A : (1, 0) \mapsto (0, -1)$ and $A : (0, 1) \mapsto (-1, 0)$. These are the columns of $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

Method 2 A can be obtained rotating by $+\pi/2$ and then reflecting across the $x-$axis:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Method 3 A can be obtained by reflecting across the $y-$axis, then rotating by $+\pi/2$:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Method 4 First reflect across the $x-$axis, then rotate by $-\pi/2$:

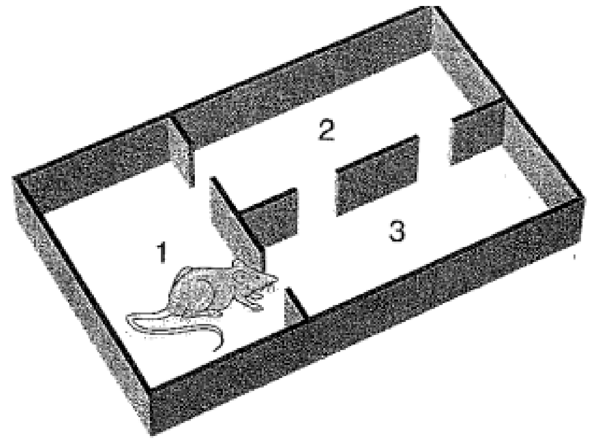
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Method 5 First rotate by $-\pi/2$, then reflect across the $y-$axis. The details should now be routine.

In summary, $TX = \begin{pmatrix} -5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} X$. As a check, $T : \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} -6 \\ -1 \end{pmatrix}$.

Note that in Methods 2–5, we just used combinations of various orthogonal transformations.

B–2. A psychologist places a rat in a cage with three compartments (see figure). The rat has been trained to select a door at random whenever a bell is rung and to move to one of the adjacent compartments.



a) If the rat is initially in compartment 1, what is the probability that it will be in compartment 2 after the bell has rung *twice*?

SOLUTION: Note that if the rat is in room 2, then, since there are two doors to compartment 3 but only one door to compartment 1, at the next stage the probability that it will be in compartment 3 is 2/3 while the probability that it will be in compartment 1 is 1/3. The transition matrix for this is

$$T := \begin{pmatrix} 0 & 1/3 & 1/3 \\ 1/2 & 0 & 2/3 \\ 1/2 & 2/3 & 0 \end{pmatrix}$$

Starting in the first compartment, after the bell has rung once and twice, the probabilities respectively are

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1/2\\1/2\end{pmatrix} \qquad \text{and} \qquad T^2\begin{pmatrix}1\\0\\0\end{pmatrix} = T\begin{pmatrix}0\\1/2\\1/2\end{pmatrix} = \begin{pmatrix}1/3\\1/3\\1/3\end{pmatrix}.$$

Thus, after the bell has rung twice, the probability is 1/3 that it will be in compartment 2 (or any of the compartments).

b) In the long run, what proportion of the time will the rat spend in each compartment?

SOLUTION: We want the stable state of this process. This is the state determined by the eigenvector V associated with the eigenvalue 1, that is, $TV = 1V$. After a straightforward computation we find $V = \begin{pmatrix}2\\3\\3\end{pmatrix}$. However, to get a probability vector we need to normalize this so the sum of the components are 1. This gives the probability eigenvector $P = \begin{pmatrix}1/4\\3/8\\3/8\end{pmatrix}$.

It tells us that the rat will spend 1/4 of its time in compartment one and 3/8 of its time in each of compartments two and three,

B–3. Let B be a diagonalizable 3×3 matrix whose rank is 1 (that is, the dimension of its image is 1). If its trace is 10, what are the eigenvalues of B? Be sure to describe any multiplicities and explain your answer.

SOLUTION: There are two approaches, one using the assumption that B can be diagonalized and one not using this assumption. [WARNING: Matrices with the same echelon form are rarely similar.]

METHOD 1 (assuming B is similar to a diagonal matrix). Since the eigenvalues and both their algebraic and geometric multiplicities (but not the eigenvectors themselves) are preserved, we can assume that B is a diagonal matrix with its eigenvalues on the diagonal:

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

The rank of a *diagonal* matrix is the number of non-zero elements on the diagonal (why?). Because B has rank 1, only one of its eigenvalues is not zero. Say $\lambda_1 \neq 0$ and $\lambda_2 = \lambda_3 = 0$. Since $10 = \operatorname{trace}(B) = \lambda_1 + \lambda_2 + \lambda_3$, we see that $\lambda_1 = 10$ with algebraic and geometric multiplicity 1 while $\lambda = 0$ has algebraic and geometric multiplicity 2.

METHOD 2 (*not* assuming B is similar to a diagonal matrix). Since $\dim(\operatorname{image}(B) = 1$, then $\dim\ker(B) = 2$ (why?). But every vector in the $\ker(B)$ is an eigenvector associated with the eigenvalue 0. Label these eigenvalues $\lambda_2 = 0$ and $\lambda_3 = 0$ and pick any two linearly independent vectors in the kernel as the eigenvectors. Since $\operatorname{trace}(B) = 10$, then $\lambda_1 = 10$ with algebraic and geometric multiplicity one.

This approach proves that a rank one matrix with a non-zero eigenvalue is similar to a diagonal matrix. The example $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ shows that a rank one matrix all of whose eigenvalues are zero might not be diagonalizable.

B–4. Let $A := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $B := \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, $C := \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

One of these can be diagonalized by an orthogonal transformation, one can be diagonalized but not by an orthogonal transformation, and one cannot be diagonalized. Identify these, explaining your reasoning.

SOLUTION: Since A is a symmetric matrix, it can be diagonalized by an orthogonal matrix.

Since $\det(B - \lambda I) = \lambda^2$, both of its eigenvalues are zero. However since $\dim\ker B = 1$, this eigenvalue has geometric multiplicity one so B can't be diagonalized.

Another approach is to use the idea of problem A-4 above to see that if all of the eigenvalues of a matrix are zero, the matrix can be diagonalized if and only if it is the 0 matrix.

Since the eigenvalues of C are distinct (they are complex conjugates), it can be diagonalized – but not by a real orthogonal matrix,

B–5. Let $A := \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ and $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$. Solve $\dfrac{d\vec{x}}{dt} = A\vec{x}$ with $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Solution: The key observation is that if A were a diagonal matrix, this would be simple. Thus we begin by finding the eigenvalues ad eigenvectors of A. By an easy calculation

$$\det(A-\lambda I)=\lambda^2-8\lambda+15=(\lambda-3)(\lambda-5).$$

Thus the eigenvalues are $\lambda_1=3$ and $\lambda_2=5$ with corresponding eigenvectors $\vec{v}_1=\begin{pmatrix}1\\-1\end{pmatrix}$ and $\vec{v}_2=\begin{pmatrix}1\\1\end{pmatrix}$. From here we can proceed in two slightly different ways.

Method 1 Observe that A is similar to the diagonal matrix $D=\begin{pmatrix}3&0\\0&5\end{pmatrix}$, that is, $S^{-1}AS=D$, where $S=\begin{pmatrix}1&1\\-1&1\end{pmatrix}$ has the corresponding eigenvectors as its columns. Thus $A=SDS^{-1}$.

We now use this in our differential equation: $\vec{x}'(t)=SDS^{-1}\vec{x}$. Multiply both sides by S^{-1}. Since S does not depend on t, $(S^{-1}\vec{x}(t))'=DS^{-1}\vec{x}$. This is simpler to use if we let $\vec{y}(t)=S^{-1}\vec{x}$. Then the differential equation becomes

$$\frac{d\vec{y}(t)}{dt}=D\vec{y}(t),$$

that is,

$$\begin{pmatrix}y_1'(t)\\y_2'(t)\end{pmatrix}=\begin{pmatrix}3&0\\0&5\end{pmatrix}\begin{pmatrix}y_1(t)\\y_2(t)\end{pmatrix}=\begin{pmatrix}3y_1(t)\\5y_2(t)\end{pmatrix}.$$

These are *uncoupled* differential equations, $y_1'=3y_1$, $y_2'=4y_2$, that one can solve immediately giving

$$y_1(t)=ae^{3t},\qquad y_2(t)=be^{5t},$$

for any constants a and b.

It remains to return to restate this in terms of $\vec{x}(t)$

$$\vec{x}(t)=S\vec{y}(t)=\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}ae^{3t}\\be^{5t}\end{pmatrix}=\begin{pmatrix}ae^{3t}+be^{5t}\\-ae^{3t}+be^{5t}\end{pmatrix}$$

We use the initial condition to determine the constants a and b.

$$\begin{pmatrix}1\\0\end{pmatrix}=\vec{x}(0)=\begin{pmatrix}a+b\\-a+b\end{pmatrix}.$$

Thus $a=b=1/2$. Therefore

$$\vec{x}(t)=\frac{1}{2}\begin{pmatrix}e^{3t}+e^{5t}\\-e^{3t}+e^{5t}\end{pmatrix}.$$

Method 2 Since the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ are a basis for $\mathbb{R}^2$, given any $x(t)$, there are functions $y_1(t)$ and $y_2(t)$ so that

$$\vec{x}(t)=y_1(t)\vec{v}_1+y_2(t)\vec{v}_2. \tag{1}$$

We now plug this in the differential equation $\vec{x}\,' = A\vec{x}$. The left side becomes

$$\vec{x}\,'(t) = y_1'(t)\vec{v}_1 + y_2'(t)\vec{v}_2,$$

and the more interesting right side becomes

$$A\vec{x} = 3y_1\vec{v}_1 + 5y_2\vec{v}_2.$$

Comparing the coefficients of $\vec{v}_1$ and $\vec{v}_2$ in the last two equations we conclude that

$$y_1' = 3y_1 \qquad \text{and} \qquad y_2' = 5y_2.$$

Their solutions are

$$y_1(t) = ae^{3t} \qquad \text{and} \qquad y_2(t) = be^{5t}$$

for any constants a and b. Using this in equation (1) we find

$$\vec{x}(t) = ae^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + be^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Finally, use the initial condition to determine a and b:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{x}(0) = a\begin{pmatrix} 1 \\ -1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This gives $a = b = 1/2$. Therefore

$$\vec{x}(t) = \frac{1}{2}\left[e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right].$$

B–6. Let A be a self-adjoint $n \times n$ matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Show that $\langle \vec{x},\, A\vec{x} \rangle \ge \lambda_1 \|\vec{x}\|^2$ for any $\vec{x}$.

SOLUTION: If A is *any* matrix with real eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, and corresponding real eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$, then $\langle \vec{v}_j,\, A\vec{v}_j \rangle = \lambda_j \|\vec{v}_j\|^2 \ge \lambda_1 \|\vec{v}_j\|^2$. This is trivial.

The point of this problem is that in the special case where A is self-adjoint, the inequality $\langle \vec{x}, A\vec{x}, \ge \rangle \lambda_1 \|\vec{x}\|^2$ holds for *any* vector $\vec{x}$.

There are two slightly different methods, identical in style to the two methods used for Problem B-5 just above.

METHOD 1. Since A is self adjoint there is an orthogonal transformation R so that $R^*AR = D$ where D is a real diagonal matrix whose diagonal elements are the eigenvalues of A. Thus $A = RDR^*$ so for any $\vec{x}$

$$\langle \vec{x},\, A\vec{x} \rangle = \langle \vec{x},\, RDR^*\vec{x} \rangle = \langle R^*\vec{x},\, DR^*\vec{x} \rangle.$$

Write $\vec{y} = R^*\vec{x}$. Then from the previous line

$$\begin{aligned}
\langle \vec{x},\, A\vec{x} \rangle &= \langle \vec{y},\, D\vec{y} \rangle \\
&= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \\
&\ge \lambda_1 (y_1^2 + \cdots + y_n^2) \\
&= \lambda_1 \|\vec{y}\|^2 = \lambda_1 \|R\vec{x}\|^2 = \lambda_1 \|\vec{x}\|^2,
\end{aligned}$$

where in the last step we used that an orthogonal transformation R preserves lengths: $\|R\vec{x}\| = \|\vec{x}\|$ for all vectors $\vec{x}$.

Method 2. Since A is self adjoint, there is an *orthonormal* basis of eigenvectors $\vec{v}_1,\ldots,\vec{v}_n$, where $A\vec{v}_j = \lambda_j\vec{v}_j$.

Any vector $\vec{x}$ can be written in this basis

$$\vec{x} = y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n. \tag{2}$$

Note that because the $\vec{v}_j$ are orthonormal,

$$\|\vec{x}\|^2 = \langle\vec{x},\, \vec{x}\rangle = y_1^2 + y_2^2 + \cdots + y_n^2. \tag{3}$$

Then

$$\begin{aligned} A\vec{x} =& A(y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n) \\ =& \lambda_1 y_1\vec{v}_1 + \lambda_2 y_2\vec{v}_2 + \cdots + \lambda_n y_n\vec{v}_n. \end{aligned}$$

Therefore, since the $\vec{v}_j$ are *orthonormal* and by equation (3)

$$\begin{aligned} \langle\vec{x},\, A\vec{x}\rangle &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \\ &\geq \lambda_1(y_1^2 + \cdots + y_n^2) = \lambda_1\|\vec{x}\|^2. \end{aligned}$$