Signature PRINTED NAME

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Math 312 **Exam 3** Jerry L. Kazdan
Dec. 6, 2012 12:00 – 1:20

DIRECTIONS This exam has two parts, Part A, shorter problems, has 5 problem (6 points each so 30 points). Part B has 6 standard problems (10 points each, so 60 points). Total is 90 points. Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. Please justify your answers with clear reasons. No credit will be given to "correct" answers with either no or incorrect reasons.

Part A: Short Problems (5 problem, 6 points each).

1. Give an example of a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ with the property that $A^4 = I$ but $A^2 \neq I$.

2. Let V and W be linear spaces and $A : V \to W$ a linear map. Show that the image of A is a linear space.

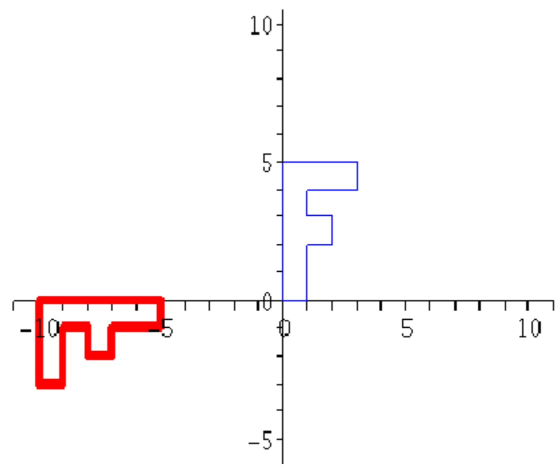
3. Let A be a square matrix. If A^2 is invertible, must A be invertible? Proof or counterexample.

4. Let A be a matrix all of whose eigenvalues are 1. If A is diagonalizable, show that A must be the identity matrix, $A = I$.

5. Let A be real matrix with a real eigenvalue λ and corresponding eigenvector $\vec{v}$. Similarly let μ be an eigenvalue of A^* with corresponding eigenvector $\vec{w}$. If $\mu \neq \lambda$, show that $\vec{v}$ and $\vec{w}$ are orthogonal.

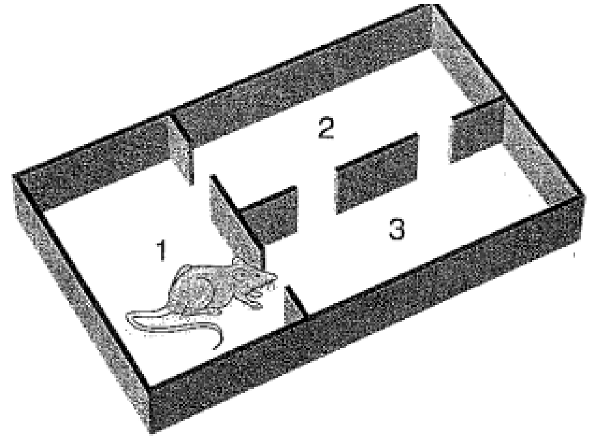
Part B: Traditional Problems (6 problems, 10 points each so 60 points)

B–1. For the following figure find a matrix A and vector V that gives the indicated transformation $TX = V + AX$. [The new **F** is bold.]



B–2. A psychologist places a rat in a cage with three compartments (see figure). The rat has been trained to select a door at random whenever a bell is rung and to move to one of the adjacent compartments.

a) If the rat is initially in compartment 1, what is the probability that it will be in compartment 2 after the bell has rung *twice*?



b) In the long run, what proportion of the time will the rat spend in each compartment?

B–3. Let B be a diagonalizable 3×3 matrix whose rank is 1 (that is, the dimension of its image is 1). If its trace is 10, what are the eigenvalues of B? Be sure to describe any multiplicities and explain your answer.

B–4. Let $A := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $B := \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, $C := \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

One of these can be diagonalized by an orthogonal transformation, one can be diagonalized but not by an orthogonal transformation, and one cannot be diagonalized. Identify these, explaining your reasoning.

B–5. Let $A := \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ and $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$. Solve $\dfrac{d\vec{x}}{dt} = A\vec{x}$ with $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

B–6. Let A be a self-adjoint $n \times n$ matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Show that $\langle \vec{x}, A\vec{x} \rangle \ge \lambda_1 \|\vec{x}\|^2$ for any $\vec{x} \ne 0$.