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Math 312
Nov. 15, 2012

Exam 2

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12:00 – 1:20

DIRECTIONS This exam has two parts. Part A has 6 shorter questions, (5 points each so total 30 points) while Part B had 5 problems (10 points each, so total is 50 points). Maximum score is thus 80 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides. *Clarity and neatness count.*

PART A: Six short answer questions (5 points each, so 30 points). To receive credit you *must* explain your reasoning at least briefly.

A–1. Find all *invertible* linear maps $A : \mathbb{R}^n \to \mathbb{R}^n$ with the property $A^2 = 3A$.

SOLUTION: Multiply both sides by A^{-1} to conclude that $A = 3I$. (You might have suspected this from the $n = 1$ case.)

A–2. Let $A : \mathbb{R}^3 \to \mathbb{R}^5$ be a linear map. If $\dim \operatorname{im}(A) = 2$, what is $\dim \operatorname{im}(A)^{\perp}$?

SOLUTION: Idea: If V is any subspace of $\mathbb{R}^n$, then $\dim V + \dim V^{\perp} = n$. Apply this to the case where $V = \operatorname{im} A \subset \mathbb{R}^5$ to find $\dim(\operatorname{im}(A))^{\perp} = 5 - \dim \operatorname{im}(A) = 3$. Notice that no fancy theorems are needed.

A–3. If a certain matrix C satisfies $\langle \vec{x}, C\vec{y} \rangle = 0$ for *all* vectors $\vec{x}$ and $\vec{y}$, show that $C = 0$.

SOLUTION: Warmup: Say that a vector $\vec{v}$ satisfies $\langle \vec{x}, \vec{v} \rangle = 0$ for *all* vectors $\vec{x}$. In particular, we can let $\vec{x} = \vec{v}$ ("$\vec{v}$ is orthogonal to itself") so we conclude that $0 = \langle \vec{v}, \vec{v} \rangle = \|\vec{v}\|^2$. Thus, $\vec{v} = 0$.

This applies to our problem with $\vec{v} = C\vec{y}$. Consequently $C\vec{y} = 0$ for all $\vec{y}$. Thus $C = 0$.

A–4. Say $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map with the property that $A^2 - 3A + 2I = 0$. If $\vec{v} \neq 0$ is an eigenvector of A with eigenvalue λ,, so $A\vec{v} = \lambda\vec{v}$, what are the possible values of λ?

SOLUTION: Since $A^2\vec{v} = A(A\vec{v}) = \lambda^2\vec{v}$, then

$$0 = (A^2 - 3A + 2I)\vec{v} = (\lambda^2 - 3\lambda + 2)\vec{v}.$$

Since $\vec{v} \neq 0$, then $\lambda^2 - 3\lambda + 2 = 0$, that is $(\lambda - 1)(\lambda - 2) = 0$ so either $\lambda = 1$ or $\lambda = 2$.

REMARK: Although $(A - I)(A - 2I) = 0$, this does not imply that either $A = I$ or $A = 2I$ as is illustrated by the example $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

A–5. Under what conditions on the constants a, b, c, and d is the following matrix A positive definite, that is, $\langle \vec{x}, A\vec{x} \rangle > 0$ for all $\vec{x} \neq 0$?

$$A := \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

SOLUTION: Say $\vec{x} = (x_1, \ldots, x_4)$. Thus, expanding we find that

$$\langle \vec{x},\, A\vec{x}\rangle = ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2 > 0$$

for all $\vec{x} \neq 0$. Letting $\vec{x} = (1,\, 0,\, 0,\, 0)$, we find that $a > 0$. Similarly, b, c, and d must all be positive.

A–6. Let A be an $n \times n$ matrix with columns A_1, A_2, $\ldots$, A_n and let B be the matrix where A_1 (the first column of A), is replaced by $3A_1 + A_2$ and the other columns are unchanged. Compute $\det B$ in terms of $\det A$.

SOLUTION: Notation: write $A = ([A_1]\,[A_2]\,\cdots\,[A_n])$. Then

$$\begin{aligned}\det B =& \det([3A_1]\,[A_2]\,\cdots,[A_n]) + \det([A_2]\,[A_2]\,\cdots\,[A_n]) \\ =& 3\det A + 0 = 3\det A\end{aligned}.$$

PART B: Five problems (10 points each, so 50 points).

B–1. Consider the space of real cubic polynomials $\mathcal{P}_3$ having degree at most 3, so $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ with the inner product $\langle f,\, g\rangle := \int_{-1}^{1} f(x)g(x)x^2\,dx$ [NOTE: This is *not* the usual inner product.]

Find the orthogonal projection of x^3 into the subspace spanned by 1 and x.

SOLUTION: Write $E = \operatorname{span}\{1, x\}$. Then we want to write

$$x^3 = a\,1 + bx + w \qquad \text{where} \qquad w \perp E. \tag{1}$$

Then the desired projection into E will be $\operatorname{proj}_E x^3 = a\,1 + bx$ while w will be the projection into $E^{\perp}$.

Note first that $1 \perp x$ since

$$\langle 1,\, x\rangle = \int_{-1}^{1} 1\,x\,x^2\,dx = \int_{-1}^{1} x^3\,dx = 0.$$

Thus, taking the inner product of both sides of (1) with 1 and then x we obtain

$$\langle x^3,\, 1\rangle = a\langle 1,\, 1\rangle \qquad \text{and} \qquad \langle x^3,\, x\rangle = b\langle x,\, x\rangle.$$

But

$$\langle x^3,\, 1\rangle = \int_{-1}^{1} x^3\,1\,x^2\,dx = 0, \qquad \langle x^3,\, x\rangle = \int_{-1}^{1} x^3\,x\,x^2\,dx = \frac{2}{7}, \qquad \langle x,\, x\rangle = \int_{-1}^{1} x\,x\,x^2\,dx = \frac{2}{5},$$

so $a = 0$ and $b = 5/7$. Consequently, $\operatorname{proj}_E(x^3) = 5x/7$.

B–2. Let A be an $n \times n$ matrix with eigenvalues λ_1, $\lambda_2, \ldots$, λ_n and corresponding eigenvectors $\vec{v}_1, \ldots$, $\vec{v}_n$. Say $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$.

a) Compute $A^2\vec{x}$ and $A^3\vec{x}$ in terms of the c_i, λ_i and $\vec{v}_i$, $i = 1, \ldots, n$.

SOLUTION: By linearity and that $A\vec{v}_i = \lambda_i\vec{v}_i$ so $A^2\vec{v}_1 = A(A\vec{v}_1) = \lambda_1^2\vec{v}_1$ etc, we have

$$\begin{aligned} A\vec{x} =& c_1 A\vec{v}_1 + \cdots + c_n A\vec{v}_n = c_1\lambda_1\vec{v}_1 + \cdots + c_n\lambda_n\vec{v}_n \\ A^2\vec{x} =& c_1\lambda_1^2\vec{v}_1 + \cdots + c_n\lambda_n^2\vec{v}_n \\ A^3\vec{x} =& c_1\lambda_1^3\vec{v}_1 + \cdots + c_n\lambda_n^3\vec{v}_n \end{aligned} .$$

b) If $\lambda_1 = 1$ and the remaining λ_j satisfy $|\lambda_j| < 1$, $j = 2, \ldots, n$, compute $\lim_{k\to\infty} A^k\vec{x}$. [This arises in the study of *Markov Chains*].

SOLUTION: Continuing, if $\lambda_1 = 1$, then

$$A^k\vec{x} = c_1\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 + \cdots + c_n\lambda_n^k\vec{v}_n.$$

But since $|\lambda_j| < 1$, $j = 2, \ldots, n$, then as $k \to \infty$ we find that $c_j\lambda_j^k \to 0$ for $j = 2, \ldots, n$. Therefore $A^k\vec{x} \to c_1\vec{v}_1$.

B–3. In an experiment, at time t you measure the value of a quantity R and obtain:

t	-1	0	1	2
R	-1	1	1	-3

Based on other information, you believe the data should fit a curve of the form $R = a + bt^2$.

a) Write the (over-determined) system of linear equations you would ideally like to solve for the unknown coefficients a and b.

SOLUTION: Using the above data, ideally we would like to solve the equations

$$\begin{aligned} a + b(-1)^2 =& -1 \\ a + b(0)^2 =& \ \ 1 \\ a + b(1)^2 =& \ \ 1 \\ a + b(2)^2 =& -3 \end{aligned}, \quad \text{that is,} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -3 \end{pmatrix}.$$

b) Use the method of least squares to find the *normal equations* for the coefficients a and b.

SOLUTION: Write the last matrix equation as $AX = Y$, then the normal equation is $A^*AX = A^*Y$ (in practice this is easier to use than the equivalent $X = (A^*A)^{-1}A^*Y$). Here

$$A^*A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 18 \end{pmatrix} \quad \text{and} \quad A^*Y = \begin{pmatrix} -2 \\ -12 \end{pmatrix}.$$

c) Solve the normal equations to find the coefficients a and b *explicitly* (numbers, like $3/5$ and -2).

SOLUTION: Explicitly, the normal equations are

$$\begin{aligned} 4a + 6b &= -2 \\ 6a + 18b &= -12 \end{aligned}.$$

The solution is $a = 1$ and $b = -1$ so our curve is $R = 1 - t^2$.

B–4. A projection $P : \mathbb{R}^n \to \mathbb{R}^n$ (so $P^2 = P$) is called an *orthogonal projection* if the image of P and kernel of P are orthogonal subspaces.

If the projection P is self-adjoint, so $P^* = P$, show that P is an orthogonal projection. [REMARK: The converse is also true: If P is an orthogonal projection, then $P = P^*$. You are not asked to prove this here.]

SOLUTION: $\vec{y}$ is in the image of P means $\vec{y} = P\vec{x}$ for some $\vec{x}$. $\vec{z}$ in in the kernel of P means $P\vec{z} = 0$. Then

$$\langle \vec{y}, \vec{z} \rangle = \langle P\vec{x}, \vec{z} \rangle = \langle \vec{x}, P^*\vec{z} \rangle = \langle \vec{x}, P\vec{z} \rangle = \langle \vec{x}, 0 \rangle = 0.$$

Thus $\vec{y}$ is orthogonal to $\vec{z}$, as desired.

Note that this did not use the property $P^2 = P$. It only used that P is self-adjoint ($P = P^*$). Thus it shows that for any self-adjoint map L, the image and kernel of L are orthogonal subspaces.

B–5. Let A be a real $n \times n$ anti-symmetric matrix, so $A^* = -A$.

a) Show that $A\vec{x}$ is orthogonal to $\vec{x}$ for *every* vector $\vec{x}$.

SOLUTION:

$$\langle A\vec{x}, \vec{x} \rangle = \langle \vec{x}, A^*\vec{x} \rangle = -\langle \vec{x}, A\vec{x} \rangle = -\langle A\vec{x}, \vec{x} \rangle.$$

Thus $2\langle A\vec{x}, \vec{x} \rangle = 0$ so $A\vec{x}$ is orthogonal to $\vec{x}$.

b) Say $\vec{x}(t)$ is a solution of the differential equation $\dfrac{d\vec{x}}{dt} = A\vec{x}$, where A is an anti-symmetric matrix. Show that $\|\vec{x}(t)\| = \text{constant}$.

SOLUTION: Idea: We show that the derivative of $\|\vec{x}(t)\|^2 = \langle \vec{x}(t), \vec{x}(t) \rangle$ is zero. For this we use the standard formula

$$\frac{d}{dt}\langle \vec{x}(t), \vec{y}(t) \rangle = \langle \vec{x}\,'(t), \vec{y}(t) \rangle + \langle \vec{x}(t), \vec{y}\,'(t) \rangle$$

in the special case when $\vec{y} = \vec{x}$. Using part a) this gives

$$\frac{d}{dt}\|\vec{x}(t)\|^2 = 2\langle \vec{x}, \vec{x}\,' \rangle = 2\langle \vec{x}, A\vec{x} \rangle = 0.$$