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GLOBAL DIFFERENTIAL GEOMETRY

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$$\int_{s_0}^{L} kx_2'\,ds = k(s_0)\int_{s_0}^{\xi_2} x_2'\,ds + k(L)\int_{\xi_2}^{L} x_2'\,ds$$
$$= x_2(\xi_2)(k(s_0) - k(0)), \qquad s_0 < \xi_2 < L.$$

Since the sum of the left-hand members is zero, these equations give

$$(x_2(\xi_1) - x_2(\xi_2))(k(0) - k(s_0)) = 0,$$

which is a contradiction, because

$$x_2(\xi_1) - x_2(\xi_2) < 0, \qquad k(0) - k(s_0) > 0.$$

It follows that there is at least one more vertex on C. Since the relative extrema occur in pairs, there are at least four vertices and the theorem is proved.

At a vertex we have $k' = 0$. Hence, we can also say that on a simple closed convex curve there are at least four points at which $k' = 0$.

The four-vertex theorem is also true for simple closed nonconvex plane curves; see:

1. S. B. Jackson, "Vertices for plane curves," *Bulletin of the American Mathematical Society* 50 (1944), pp. 564–578.
2. L. Vietoris, "Ein einfacher Beweis des Vierscheitelsatzes der ebenen Kurven," *Archiv der Mathematik* 3 (1952), pp. 304–306.

For further reading, see:

1. P. Scherk, "The four-vertex theorem," *Proceedings of the First Canadian Mathematical Congress*. Montreal: 1945, pp. 97–102.

3. ISOPERIMETRIC INEQUALITY FOR PLANE CURVES

The theorem can be stated as follows.

THEOREM: *Among all simple closed curves having a given length the circle bounds the largest area. In other words, if L is the length of a simple closed curve C, and A is the area it bounds, then*

$$L^2 - 4\pi A \geqq 0. \tag{7}$$

Moreover, the equality sign holds only when C is a circle.

Many proofs have been given of this theorem, differing in degree of elegance and in the range of curves under consideration—that is, whether differentiability or convexity is supposed. We shall give two proofs, the work of E. Schmidt (1939) and A. Hurwitz (1902), respectively.

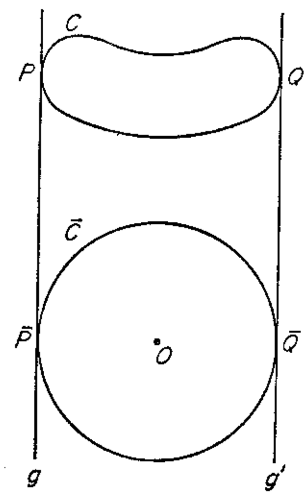

FIG. 3

Schmidt's Proof: We enclose C between two parallel lines, g and g', such that C lies between g and g' and is tangent to them at the points P and Q, respectively. We let $s = 0$, s_0 being the parameters of P and Q, and construct a circle $\bar{C}$ tangent to g and g' at $\bar{P}$ and $\bar{Q}$, respectively. Denote its radius by r and take its center to be the origin of a coordinate system. Let $X(s) = (x_1(s), x_2(s))$ be the position vector of C, so that $(x_1(0), x_2(0)) = (x_1(L), x_2(L))$. As the position vector of $\bar{C}$ we take $(\bar{x}_1(s), x_2(s))$, such that

$$\begin{aligned} \bar{x}_1(s) &= x_1(s), \\ \bar{x}_2(s) &= -\sqrt{r^2 - x_1^2(s)}, \quad 0 \leqq s \leqq s_0 \\ &= +\sqrt{r^2 - x_1^2(s)}, \quad s_0 \leqq s \leqq L. \end{aligned} \tag{8}$$

Denote by $\bar{A}$ the area bounded by $\bar{C}$. Now the area bounded by a closed curve can be expressed by the line integral

$$A = \int_0^L x_1x_2'\,ds = -\int_0^L x_2x_1'\,ds = \tfrac{1}{2}\int_0^L (x_1x_2' - x_2x_1')\,ds.$$

Applying this to our two curves C and $\bar{C}$, we get

$$A = \int_0^L x_1x_2'\,ds$$

$$\bar{A} = \pi r^2 = -\int_0^L \bar{x}_2\bar{x}_1'\,ds = -\int_0^L \bar{x}_2x_1'\,ds.$$

Adding these two equations, we have

$$A + \pi r^2 = \int_0^L (x_1 x_2' - \bar{x}_2 x_1')\, ds \leqq \int_0^L \sqrt{(x_1 x_2' - \bar{x}_2 x_1')^2}\, ds$$

$$(9) \qquad \leqq \int_0^L \sqrt{(x_1^2 + \bar{x}_2^2)(x_1'^2 + x_2'^2)}\, ds$$

$$= \int_0^L \sqrt{x_1^2 + \bar{x}_2^2}\, ds = Lr.$$

Since the geometric mean of two positive numbers is less than or equal to their arithmetic mean, it follows that

$$\sqrt{A}\,\sqrt{\pi r^2} \leqq \tfrac{1}{2}(A + \pi r^2) \leqq \tfrac{1}{2}Lr,$$

which gives, after squaring and cancellation of r^2, the inequality in Equation (7).

Suppose now that the equality sign in Equation (7) holds; then A and πr^2 have the same geometric and arithmetic mean, so that $A = \pi r^2$ and $L = 2\pi r$. The direction of the lines g and g' being arbitrary, this means that C has the same "width" in all directions. Moreover, we must have the equality sign everywhere in Equation (9). It follows, in particular, that

$$(x_1 x_2' - \bar{x}_2 x_1')^2 = (x_1^2 + \bar{x}_2^2)(x_1'^2 + x_2'^2),$$

which gives

$$\frac{x_1}{x_2'} = \frac{-\bar{x}_2}{x_1'} = \frac{\sqrt{x_1^2 + \bar{x}_2^2}}{\sqrt{x_1'^2 + x_2'^2}} = \pm r.$$

From the first equality in Equation (9), the factor of proportionality is seen to be r, that is,

$$x_1 = r x_2', \qquad \bar{x}_2 = -r x_1',$$

which remains true when we interchange x_1 and x_2, so that

$$x_2 = r x_1'.$$

Therefore, we have

$$x_1^2 + x_2^2 = r^2,$$

which means that C is a circle.

Hurwitz's proof makes use of the theory of Fourier series. We shall first prove the lemma of Wirtinger.

LEMMA: *Let $f(t)$ be a continuous periodic function of period 2π, possessing a continuous derivative $f'(t)$. If $\int_0^{2\pi} f(t)\, dt = 0$, then*

$$(10) \qquad \int_0^{2\pi} f'(t)^2\, dt \geqq \int_0^{2\pi} f(t)^2\, dt.$$

Moreover, the equality sign holds if and only if

$$(11) \qquad f(t) = a \cos t + b \sin t.$$

Proof: To prove the lemma, we let the Fourier series expansion of $f(t)$ be

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

Since $f'(t)$ is continuous, its Fourier series can be obtained by differentiation term by term, and we have

$$f'(t) \sim \sum_{n=1}^{\infty} (n b_n \cos nt - n a_n \sin nt).$$

Since

$$\int_0^{2\pi} f(t)\, dt = \pi a_0,$$

it follows from our hypothesis that $a_0 = 0$. By Parseval's formula, we get

$$\int_0^{2\pi} f(t)^2\, dt = \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

$$\int_0^{2\pi} f'(t)^2\, dt = \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2).$$

Hence,

$$\int_0^{2\pi} f'(t)^2\, dt - \int_0^{2\pi} f(t)^2\, dt = \sum_{n=1}^{\infty} (n^2 - 1)(a_n^2 + b_n^2),$$

which is greater than, or equal to, 0. It is equal to zero, only if $a_n = b_n = 0$ for all $n > 1$. Therefore, $f(t) = a_1 \cos t + b_1 \sin t$, which proves the lemma.

Hurwitz's Proof: In order to prove the inequality in Equation (7), we assume, for simplicity, that $L = 2\pi$, and that

$$\int_0^{2\pi} x_1(s)\, ds = 0.$$

The latter means that the center of gravity lies on the x_1-axis, a condition which can always be achieved by a proper choice of the coordinate system. The length and the area are given by the integrals,

$$2\pi = \int_0^{2\pi} (x_1'^2 + x_2'^2)\, ds, \quad \text{and} \quad A = \int_0^{2\pi} x_1 x_2'\, ds.$$

From these two equations we get

$$2(\pi - A) = \int_0^{2\pi} (x_1'^2 - x_1^2)\, ds + \int_0^{2\pi} (x_1 - x_2')^2\, ds.$$

The first integral is greater than, or equal to, 0 by our lemma and the second integral is clearly greater than, or equal to, 0. Hence, $A \leqq \pi$, which is our isoperimetric inequality.

The equality sign holds only when

$$x_1 = a \cos s + b \sin s, \qquad x_2' = x_1,$$

which gives

$$x_1 = a \cos s + b \sin s, \qquad x_2 = a \sin s - b \cos s + c.$$

Thus, C is a circle.

For further reading, see:

1. E. Schmidt, "Beweis der isoperimetrischen Eigenschaft der Kugel im hyperbolischen und sphärischen Raum jeder Dimensionenzahl," *Math. Zeit.* 49 (1943), pp. 1–109.

4. TOTAL CURVATURE OF A SPACE CURVE

The *total curvature* of a closed space curve C of length L is defined by the integral

$$\mu = \int_0^L |k(s)|\, ds, \tag{12}$$

where $k(s)$ is the curvature. For a space curve, only $|k(s)|$ is defined.

Suppose C is oriented. Through the origin O of our space we draw vectors of length 1 parallel to the tangent vectors of C. Their end-points describe a closed curve Γ on tne unit sphere, to be called the *tangent indicatrix* of C. A point of Γ is singular (that is, with either no tangent or a tangent of higher contact) if it is the image of a point of zero curvature of C. Clearly the total curvature of C is equal to the length of Γ.

Fenchel's theorem concerns the total curvature.

THEOREM: *The total curvature of a closed space curve C is greater than, or equal to, 2π. It is equal to 2π if and only if C is a plane convex curve.*

The following proof of this theorem was found independently by B. Segre (*Bolletino della Unione Matematica Italiana* 13 (1934), 279–283), and by H. Rutishauser and H. Samelson (*Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences* 227 (1948), 755–757). See also W. Fenchel, *Bulletin of the American Mathematical Society* 57 (1951), 44–54. The proof depends on the following lemma:

LEMMA: *Let Γ be a closed rectifiable curve on the unit sphere, with length $L < 2\pi$. There exists a point m on the sphere such that the spherical distance $\overline{mx} \leqq L/4$ for all points x of Γ. If Γ is of length 2π but is not the union of two great semicircular arcs, there exists a point m such that $\overline{mx} < \pi/2$ for all x of Γ.*

We use the notation $\overline{ab}$ to denote the spherical distance of two points, a and b. If $\overline{ab} < \pi$, their *midpoint* m is the point defined by the conditions $\overline{am} = \overline{bm} = \frac{1}{2}\overline{ab}$. Let x be a point such that $\overline{mx} \leqq \frac{1}{2}\pi$. Then $2\overline{mx} \leqq \overline{ax} + \overline{bx}$. In fact, let x' be the symmetry of x relative to m. Then,

$$\overline{x'a} = \overline{xb}, \qquad \overline{x'x} = \overline{x'm} + \overline{mx} = 2\overline{mx}.$$

If we use the triangle inequality, it follows that

$$2\overline{mx} = \overline{x'x} \leqq \overline{x'a} + \overline{ax} = \overline{ax} + \overline{bx}, \tag{13}$$

as to be proved.

Lemma Proof: To prove the first part of the lemma, we take two points, a and b, on Γ which divide the curve into two equal arcs. Then $\overline{ab} < \pi$, and we denote the midpoint by m. Let x be a point of Γ such that $2\overline{mx} < \pi$. Such points exist—for example, the point a. Then we have

$$\overline{ax} \leqq \widehat{ax}, \qquad \overline{bx} \leqq \widehat{bx},$$

where $\widehat{ax}$ and $\widehat{bx}$ are respectively the arc lengths along Γ. From Equation (13), it follows that

$$2\overline{mx} \leqq \widehat{ax} + \widehat{bx} = \widehat{ab} = \frac{L}{2}.$$

Hence, the function $f(x) = \overline{mx}$, $x \in \Gamma$, is either $\geqq \pi/2$ or $\leqq L/4 < \pi/2$. Since Γ is connected and $f(x)$ is a continuous function in Γ, the range of the function $f(x)$ is connected in the interval $(0, \pi)$. Therefore, we have $f(x) = \overline{mx} \leqq L/4$.

Consider next the case that Γ is of length 2π. If Γ contains a pair of antipodal points, then, being of length 2π, it must be the union of two great semicircular arcs. Suppose that there is a pair of points, a and b, which bisect Γ such that

$$\overline{ax} + \overline{bx} < \pi$$

for all $x \in \Gamma$. Again, let m denote the midpoint of a and b. If $f(x) = \overline{mx} \leqq \frac{1}{2}\pi$, we have, from Equation (13),

$$2\overline{mx} \leqq \overline{ax} + \overline{bx} < \pi,$$

which means that $f(x)$ omits the value $\pi/2$. Since its range is connected and since $f(a) < \pi/2$, we have $f(x) < \pi/2$ for all $x \in \Gamma$. Thus the lemma is true in this case.

It remains to consider the case that Γ contains no pair of antipodal points, and that for any pair of points a and b which bisect Γ, there is a point $x \in \Gamma$ with

$$\overline{ax} + \overline{bx} = \pi.$$

An elementary geometrical argument, which we leave to the reader, will show that this is impossible. Thus, the lemma is proved.

Theorem Proof: To prove Fenchel's theorem, we take a fixed unit vector A and put

$$g(s) = AX(s),$$

where the right-hand side denotes the scalar product of the vectors A and $X(s)$. The function $g(s)$ is continuous on C and hence must have a maximum and a minimum. Since $g'(s)$ exists, we have, at such an extremum s_0,

$$g'(s_0) = AX'(s_0) = 0.$$

Thus A, as a point on the unit sphere, has a distance $\pi/2$ from at least two points of the tangent indicatrix. Since A is arbitrary, the tangent indicatrix is met by every great circle. It follows from the lemma that its length is greater than, or equal to, 2π.

Suppose next that the tangent indicatrix Γ is of length 2π. By our lemma, it must be the union of two great semicircular arcs. It follows that C itself is the union of two plane arcs. Since C has a tangent everywhere, it must be a plane curve. Suppose C be so oriented that its rotation index

$$\frac{1}{2\pi}\int_0^L k\,ds \geqq 0.$$

Then we have

$$0 \leqq \int_0^L \{|k| - k\}\,ds = 2\pi - \int_0^L k\,ds$$

so that the rotation index is either 0 or 1. To a given vector in the plane there is parallel to it a tangent t of C such that C lies to the left of t. Then t is parallel to the vector in the same sense, and at its point of contact we have $k \geqq 0$, implying that $\int_{k>0} k\,ds \geqq 2\pi$. Since $\int_C |k|\,ds = 2\pi$, there is no point with $k < 0$, and $\int k\,ds = 2\pi$. From the remark at the end of Section 1, we conclude that C is convex.

As a corollary we have the following theorem.

COROLLARY: *If $|k(s)| \leqq 1/R$ for a closed space curve C, C has a length $L \geqq 2\pi R$.*

We have

$$L = \int_0^L ds \geqq \int_0^L R|k|\,ds = R\int_0^L |k|\,ds \geqq 2\pi R.$$

Fenchel's theorem holds also for sectionally smooth curves. As the total curvature of such a curve we define

$$\mu = \int_0^L |k|\,ds + \sum_i a_i \tag{14}$$

where the a_i are the angles at the vertices. In other words, in this