Problem Set 9

DUE: Thurs. March 22. *Late papers will be accepted until 1:00 PM Friday.*

Unless otherwise stated use the standard Euclidean *norm.*

1. a) Find the local maxima and minima of $f(x,y) := 3x + 4y$ for $x^2 + y^2 < 1$.
   b) Find the maxima and minima of $f(x,y) := 3x + 4y$ for $x^2 + y^2 = 1$.
   c) Find the global maxima and minima of $f(x,y) := 3x + 4y$ for $x^2 + y^2 \le 1$.

2. a) Let x, y, and z be positive numbers such that $x + y + z = a$, where a is a fixed positive number. How large can their product, xyz, be?
   b) If x, y, and z are nonnegative, show that
$$(xyz)^{1/3} \le \frac{x+y+z}{3}.$$

3. Let $f(x,y) := [x^2 + (y+2)^2][x^2 + (y-2)^2]$.
   a) Find and classify all of the critical points of f in $\mathbb{R}^2$.
   b) Find the maximum and minimum value of f on the circle $x^2 + y^2 = 1$.
   c) Find the global maximum and minimum value of f in the *closed* disk $x^2 + y^2 \le 1$.
   d) Find the global maximum and minimum value of f in the *closed* disk $x^2 + y^2 \le 9$.

   NOTE: In Maple
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w:= (x,y) -> (x^2 + (y+2)^2)*(x^2 + (y-2)^2);
with(plots):
plot3d(w(x,y), x=-2..2, y=-3.2..3.2, view=-0.5..50, axes=normal);
```

4. Find the maximum and minimum value of $h(x,y,z) := (x-y)^2 + z^2$ on $x^2 + y^2 + z^2 = 18$.

5. In the plane $\mathbb{R}^2$, let $r^2 = x^2 + y^2$ so r is the distance from the origin to (x, y) and assume that the function $u(x,y)$ depends only on r. Thus $u(x,y) = w(r)$ for some (smooth) function w.
   a) Show that $\dfrac{\partial u}{\partial x} = \dfrac{dw}{dr}\dfrac{x}{r}$.
   b) Show that
$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 w}{dr^2}\left(\frac{x^2}{r^2}\right) + \left(\frac{1}{r} - \frac{x^2}{r^3}\right)\frac{dw}{dr}.$$
   c) Show that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}.$$

d) Find all functions $u(x,y)$ depending only on r that satisfy the *Laplace equation* $\Delta u := u_{xx} + u_{yy} = 0$. [Physicists often write Δu as $\nabla^2 u$ or $\nabla \cdot \nabla u$].

e) Generalize the previous parts to (x,y,z) in $\mathbb{R}^3$. Here $r^2 = x^2 + y^2 + z^2$ and the Laplacian equation is $\Delta u := u_{xx} + u_{yy} + u_{zz} = 0$.

6. Let Ω be the rectangle with vertices at $(-1,0)$, $(1,0)$, $(1,2)$, and (-1.2). Estimate the value of both of the following integrals.

$$a). \quad \iint_\Omega (x^2 + y^2)\,dA \qquad\qquad b). \quad \iint_\Omega x(1-2y)\,dA$$

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Find the *exact* volume of the solid region bounded from below by $z := f(x,y) - 1$ and bounded from above by $z := f(x,y) + 3$ for $-1 \le x \le 5$ and $-2 \le y \le 2$.

8. Let $J := \iint_\Omega (x-2y)^2\,dA$ where Ω is the triangular region bounded by the lines $x = 1$, $y = -2$, and $y + 2x = 4$.

   a) Evaluate this by integrating first with respect to x, so $J = \int_?^? \left(\int_?^? (x-2y)^2\,dx \right) dy$.

   b) Evaluate this by integrating first with respect to y.

9. Let $K := \int_0^1 \left(\int_0^{2x} f(x,y)\,dy \right) dx$.

   a) Draw a sketch of the region of integration $\Omega \subset \mathbb{R}^2$.

   b) Find the limits of integration if you integrate first with respect to x, so $K = \int_?^? \left(\int_?^? f(x,y)\,dx \right) dy$.

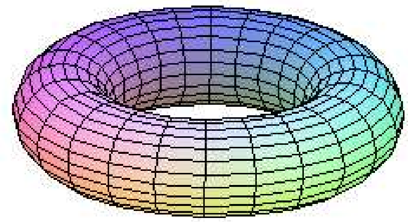
Bonus Problems

[Please give these directly to Professor Kazdan]

B-1 Let A be a real symmetric $n \times n$ matrix. If V is a vector that minimizes the quadratic polynomial $\langle X, AX \rangle$ on the unit sphere $\|X\|^2 = 1$, show there is a number λ so that $AV = \lambda V$. In other words, V is an eigenvector of A with eigenvalue λ. [This is the idea behind the standard procedure physicists use to compute the lowest energy level for a solution of the Schrödinger equation.]

B-2 It is often useful to define surfaces using *parametric equations*. For instance, a torus is most simply obtained by rotating a circle, say a unit circle centered at $(3,0)$ in the r,z plane, $r(\phi) := 3+\cos\phi,\ z(\phi) = \sin\phi$, around the (vertical) z axis, so:

$$
\begin{aligned}
x(\theta,\phi) &:= (3+\cos\phi)\cos\theta\\
y(\theta,\phi) &:= (3+\cos\phi)\sin\theta\\
z(\theta,\phi) &:= \sin\phi.
\end{aligned}
$$



Here the parameters are θ and ϕ.

Maple:

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with(plots):
a:=3; b:=1;
r:= phi -> a + b*cos(phi);
plot3d([r(phi)*cos(theta), r(phi)*sin(theta), b*sin(phi)],
phi=0..2*Pi, theta=0..2*Pi, scaling=constrained);
```

Find the equation of the tangent plane at $\theta = 0$, $\phi = \pi/4$. [Before computing, use the figure to guess what it should be.]

[Last revised: March 13, 2012]