Problem Set 7

DUE: In class Thursday, March 1. *Late papers will be accepted until 1:00 PM Friday.*

Unless otherwise stated use the standard Euclidean *norm.*

1. Say you have a matrix $A(t) = ((a_{ij}(t)))$ whose elements $a_{ij}(t)$ depend on a parameter $t \in \mathbb{R}$. We say that $A(t)$ is differentiable at t_0 if the limit

$$\lim_{h \to 0} \frac{A(t_0 + h) - A(t_0)}{h}$$

exists. We then call this limit $A'(t_0)$. It is easy to show that $A(t)$ is differentiable if and only if all of the elements $a_{ij}(t)$ are differentiable.

   a) Say $A(t)$ and $B(t)$ are $n \times n$ matrices that are differentiable and let $C(t) := A(t)B(t)$ Show that $C(t)$ is differentiable and give a formula for $C'(t)$ in terms of A, A', B, and B'

   b) If $A(t)$ is differentiable and invertible, show that $A^{-1}(t)$ is differentiable and give a formula for $dA^{-1}(t)/dt$ in terms of A, A^{-1}, and A'. [SUGGESTION: Imitate the proof for a 1×1 matrix.]

2. In each of the following find $u(x, t)$.

   a) $\dfrac{\partial u}{\partial t} = 0 \qquad \text{with} \qquad u(x, 0) = e^x \sin 2x$.

   b) $\dfrac{\partial u}{\partial t} = 2t \qquad \text{with} \qquad u(x, 0) = e^x \sin 2x$.

   c) $\dfrac{\partial u}{\partial t} + u = 0 \qquad \text{with} \qquad u(x, 0) = e^x \sin 2x$.

3. a) If $u(x, t)$ satisfies $u_t - 2u_x = 0$ with $u(x, 0) = 7 + 2 \sin x$, find $u(x, t)$. [SUGGESTION: Intrepret the differential equation as stating that the directional derivative in a certain direction is zero.]

   b) Find $v(x, t)$ with the properties $v_t - 2v_x = 2t$ with $v(x, 0) = 7 + 2 \sin x$.

4. Let R be an $n \times n$ matrix with the property that

$$\langle RX,\, RY \rangle = \langle X,\, Y \rangle \qquad \text{for all} \quad X,\, Y \in \mathbb{R}^n, \tag{1}$$

so R preserves the inner product and hence the lengths of vectors and the angles between vectors.

   a) Show that $R^*R = I$, so $R^* = R^{-1}$.

   b) Conversely, if $R^* = R^{-1}$, show that (1) is satisfied.

c) Letting $Y = X$ in equation (1) shows that R preserves the lengths of vectors: $\|RX\| = \|X\|$ for all X. Conversely, if R preserves the lengths of vectors, show that equation (1) is true.
[SUGGESTION: Prove and use the identity $4\langle X, Y, =\rangle \|X+Y\|^2 - \|X-Y\|^2$]

5. [Marsden-Tromba Sec. 3.1 #11] Show that the following functions satisfy the *one dimensional wave equation*

$$u_{tt} = c^2 u_{xx}.$$

a) $u(x,t) = \sin(x - ct)$.
b) $u(x,t) = (\sin x)(\sin ct)$.
c) $u(x,t) = (x-ct)^6 + (x+ct)^6$.

6. [Marsden-Tromba, Sec 3.1, p.157 #27] A function $u(x,y,z)$ is called *harmonic* if it satisfies the homogeneous equation $u_{xx} + u_{yy} + u_{zz} = 0$. For which constant(s) a and b is $u(x,y,z) := x^2 + ay^2 + z^2 + byz - 7z$ a harmonic function?

7. [Marsden-Tromba, Sec 3.1, p.158 #31] Show that the *Newtonian gravitational potential* $v(x,y,z) := -GmM/r$ is a harmonic function – except at the origin (see the previous problem). Here G, m, and M are constants and $r^2 = x^2 + y^2 + z^2$

8. Assume the real-valued functions $u(x,y)$ and $v(x,y)$ satisfy the *Cauchy-Riemann equations*

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

Show that both u and v are harmonic functions (in two variables), that is they satisfy $\varphi_{xx} + \varphi_{yy} = 0$.

9. [Marsden-Tromba, Sec 3.2, p.166 #9] Calculate the second-order Taylor approximation to $f(x,y) := x \cos \pi y + y \sin \pi x$ at the point $(1, 2)$.

10. Find all the critical points of $f(x,y) = \sin x \sin y$ and classify them as local maxima, minima, and saddle points.

11. Let $f(x,y) := (x^2 + 4y^2)e^{(1-x^2-y^2)}$. Find and classify all of its critical points. [See the Maple worksheet

`http://www.math.upenn.edu/~kazdan/260S12/notes/volcano-Maple15.mw`

for a similar function].

12. [Marsden-Tromba, Sec 3.3, p.184 #46] A smooth function $u(x, y)$ *strictly subharmonic* if it satisfies $u_{xx} + u_{yy} > 0$. Let $u(x, y)$ be a strictly subharmonic function in the unit disk, $x^2 + y^2 < 1$. Show that u cannot have a local maximum at an interior point of this disk.

Bonus Problem

[Please give these directly to Professor Kazdan]

B-1 Let $w(x)$ and $u(x, y)$ be given smooth functions.

a) If w satisfies $w'' - c(x)w = 0$, where $c(x) > 0$ is a given function, show that w cannot have a local positive maximum. Also show that w cannot have a local negative minimum.

b) If u satisfies $4u_{xx} + 3u_{yy} - 5u = 0$, show that it cannot have a local positive maximum. Also show that u cannot have a local negative minimum.

c) Repeat the above for a solution of $4u_{xx} - 2u_{xy} + 3u_{yy} + 7u_x + u_y - 5u = 0$.

d) If a function $u(x, y)$ satisfies the above equation in a bounded region $\mathcal{D} \in \mathbb{R}^2$ and is zero on the boundary of the region, show that $u(x, y)$ is zero throughout the region.

[Last revised: February 26, 2012]