Problem Set 6

DUE: In class Thursday, Feb. 23. *Late papers will be accepted until 1:00 PM Friday.*

Unless otherwise stated use the standard Euclidean *norm.*

1. Let $u(x,t)$ be the temperature at time t at a point x on a homogeneous rod of length π, say $0 \le x \le \pi$. Assume u satisfies the *heat equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions

$$u(0,t) = 0, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=\pi} = 0 \quad \text{(so the right end is insulated)}$$

and initial condition

$$u(x,0) = \sin \tfrac{5}{2}x.$$

   a) Find the solution.
   b) Use $Q(t) = \frac{1}{2}\int_0^\pi u^2(x,t)\,dx$ to prove there is at most one solution (uniqueness).

2. [Marsden-Tromba Sec. 2.4#1-2] Sketch the curves

   a). $x = \sin t,\ y = 4\cos t,\ 0 \le t \le 2\pi$ *b*). $x = 2\sin t,\ y = 4\cos t,\ 0 \le t \le 2\pi$

3. Consider the curve $F(t) = (\sin 2t, 1 - 3t, \cos 2t,\ 2t^{3/2})$ for $0 \le t \le 2\pi$.
   a) Find the equation of the tangent line at $t = \pi/2$.
   b) Find the length of this curve.

4. [Marsden-Tromba Sec. 2.4#20] Suppose a particle follows the path $\mathbf{c}(t) = (e^t,\ e^{-t},\ \cos \pi t)$ and flies off on the tangent at $t = 1$. What is its position at $t = 2$?

5. [Marsden-Tromba Sec. 4.2#17] Say a path $X(s)$ is parametrized by *arc length* s, so $\|X'(s)\| = 1$. The *curvature*, $k(s)$, at a point $X(s)$ is defined by $k(s) := \|\frac{d^2X(s)}{ds^2}\|$. Calculate the curvature of the following curves. Note that for the first step you will need to find the arc length function $s(t)$.
   a) Circle of radius r: $X(t) := (r\cos t,\ r\sin t)$.
   b) Helix: $H(t) := \frac{1}{\sqrt{2}}(\cos t,\ \sin t,\ ct)$, where c is a constant.

6. Let $X(t): \mathbb{R}^1 \to \mathbb{R}^3$ be a smooth curve with the property that $X''(t) = 0$ for all t. What can you conclude? Prove your assertion.

7. Let $X(t) : \mathbb{R}^1 \to \mathbb{R}^3$ be a twice differentiable function that satisfies the ordinary differential equation

$$X'' + \mu X' + kX = 0, \tag{1}$$

where k and μ are *positive* constants. Define the "energy" as

$$E(t) := \tfrac{1}{2}[\|X'\|^2 + k\|X\|^2,$$

where we use the usual Euclidean norm in $\mathbb{R}^3$.

a) Show that $dE/dt \le 0$.

b) Show that there is at most one solution of the ODE (1) with initial conditions $X(0) = A$, and $X'(0) = B$, where A and B are given vectors in $\mathbb{R}^3$.

Bonus Problem

[Please give these directly to Professor Kazdan]

B-1 Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. If A is not one-to-one, but the equation $Ax = y$ has some solution, then it has many. Is there a "best" possible answer? What can one say? Think about this before reading the next paragraph.

If A is onto, so there is some solution of $Ax = y$, show there is exactly one solution x_1 of the form $x_1 = A^*w$ for some w, so $AA^*w = y$. Moreover of all the solutions x of $Ax = y$, show that x_1 is closest to the origin (in the Euclidean distance). [Remark: This situation is related to the case where where A is not onto, so there may not be a solution — but the method of least squares gives an "best" approximation to a solution.]

B-2 Let $P_1, P_2, \ldots, P_k$ be k points (think of them as *data*) in $\mathbb{R}^3$ and let $\mathcal{S}$ be the plane

$$\mathcal{S} := \left\{ X \in \mathbb{R}^3 : \langle X, N\rangle = c \right\},$$

where $N \neq 0$ is a unit vector normal to the plane and c is a real constant.

This problem outlines how to find the plane that *best approximates the data points* in the sense that it minimizes the function

$$Q(N, c) := \sum_{j=1}^{k} \text{distance}\,(P_j, \mathcal{S})^2.$$

Determining this plane means finding N and c.

a) Show that for a given point P, then

$$\text{distance}\,(P, \mathcal{S}) = |\langle P - X,\, N\rangle| = |\langle P,\, N\rangle - c\,|,$$

where X is any point in $\mathcal{S}$

b) First do the special case where the center of mass $\bar{P} := \frac{1}{k}\sum_{j=1}^{k} P_j$ is at the origin, so $\bar{P} = 0$. Show that for any P, then $\langle P,\, N\rangle^2 = \langle N,\, PP^*N\rangle$. Here view P as a column vector so PP^* is a $k \times k$ matrix.

Use this to observe that the desired plane $\mathcal{S}$ is determined by letting N be an eigenvector of the matrix

$$A := \sum_{j=1}^{k} P_j P_j^T$$

corresponding to it's lowest eigenvalue. What is c in this case?

c) Reduce the general case to the previous case by letting $V_j = P_j - \bar{P}$.

d) Find the equation of the line $ax + by = c$ that, in the above sense, best fits the data points $(-1, 3)$, $(0, 1)$, $(1, -1)$, $(2, -3)$.

e) Let $P_j := (p_{j1}, \ldots, p_{j3})$, $j = 1, \ldots, k$ be the coordinates of the j^{th} data point and $Z_\ell := (p_{1\ell}, \ldots, p_{k\ell})$, $\ell = 1, \ldots, 3$ be the vector of ℓ^{th} coordinates. If a_{ij} is the ij element of A, show that $a_{ij} = \langle Z_i,\, Z_j\rangle$. This exhibits A as a *Gram matrix*.

f) Generalize to where $P_1,\, P_2, \ldots,\, P_k$ are k points in $\mathbb{R}^n$.

[Last revised: March 6, 2012]