Problem Set 5

DUE: In class Thursday, Feb. 16. *Late papers will be accepted until 1:00 PM Friday.*

Unless otherwise stated use the standard Euclidean *norm.*

1. a) Let $f(x) = \begin{cases} -1 & \text{for } -\pi \le x < 0, \\ 1 & \text{for } 0 \le x < \pi \end{cases}$. Find its Fourier series (either using trig functions or the complex exponential).

   b) There is an extension of the Pythagorean Theorem that expresses $\|f\|^2$ in terms of the Fourier coefficients. What does that assert for this function?

   c) Let $g(x) = \begin{cases} 0 & \text{for } -\pi \le x < 0, \\ 1 & \text{for } 0 \le x < \pi \end{cases}$. Find its Fourier series.

2. For complex vectors $Z = (z_1, z_2, \ldots, z_n)$ recall that we used the inner product

$$\langle Z, W\rangle := z_1\bar{w}_1 + \cdots z_n\bar{w}_n.$$

   Say you have an $n\times n$ matrix $A := (a_{jk})$ whose elements a_{jk} may be complex numbers. Define its *adjoint* A^* by the rule $\langle A^*Z, W\rangle = \langle Z, AW\rangle$ for all complex vectors Z, W. Find a formula for the elements of A^* in terms of the elements of A. [First do the 1×1 and 2×2 casesz.]

3. A square real matrix is called *symmetric* (or *self-adjoint*) if $A = A^*$. It is called *anti-symmetric* (or *skew-adjoint*) if $A = -A^*$.

   a) If a real matrix is anti-symmetric, show that the elements on its diagonal must all be zero.

   b) If A is a square real matrix, shot there is a unique symmetric matrix A_+ and a unique anti-symmetric matrix A_- such that $A = A_+ + A_-$. [Find formulas for A_+ and A_- in terms of A and A^*.]

   c) Find a symmetric 3×3 matrix A so that

$$\langle X, AX\rangle = 3x_1^2 + 4x_1x_2 - x_2^2 - x_2x_3.$$

   SUGGESTION: First do the simpler case of finding a 2×2 matrix A so that

$$\langle X, AX\rangle = 3x_1^2 + 4x_1x_2 - x_2^2.$$

   A simple but useful observation is that $4x_1x_2 = 2x_1x_2 + 2x_2x_1$.

   d) If S is anti-symmetric, show that $\langle X, SX\rangle = 0$ for all real vectors X. MORAL: In considering the quadratic polynomial $\langle X, AX\rangle$ we always take A to be symmetric.

4. [Completing the Square] Let

$$\begin{aligned} Q(x) &= \sum a_{ij}x_ix_j + \sum b_ix_i + c \\ &= \langle x,\, Ax\rangle + \langle b,\, x\rangle + c \end{aligned}$$

be a real quadratic polynomial so $x = (x_1, \ldots, x_n)$, $b = (b_1, \ldots, b_n)$ are real vectors and $A = a_{ij}$ is a real symmetric $n \times n$ matrix. Just as in the case $n = 1$, if A is invertible show there is a change of variables $y == x - v$ (this is a translation by the vector v) so that in the new y variables Q has the form

$$\hat{Q}(y) := Q(y+v) = \langle y,\, Ay\rangle + \gamma \quad \text{that is,} \quad \hat{Q}(y) = \sum a_{ij}y_iy_j + \gamma,$$

where γ involves A, b, and c – but no terms that are linear in y. [In the case $n = 1$, which you should try *first*, this means using a change of variables $y = x - v$ to change the polynomial $ax^2 + bx + c$ to the simpler $ay^2 + \gamma$.]

As an example, apply this to $Q(x) = 2x_1^2 + 2x_1x_2 + 3x_2 - 4$.

5. We can define the derivative of a vector $X(t)$ that depends on a real parameter t as the usual limit

$$\frac{dX(t)}{dt} = \lim_{h\to 0} \frac{X(t+h) - X(t)}{h}.$$

It is easy to see that this is the same as differentiating each component of the vector. If $X(t)$ gives the path of a particle at time t, then $X'(t)$ is the *velocity vector*. It is tangent to the path.

   a) Show that the following product rule holds:

$$\frac{d}{dt}\langle X(t),\, Y(t)\rangle = \langle X'(t),\, Y(t)\rangle + \langle X(t),\, Y'(t)\rangle.$$

   b) Let Z be a fixed vector (not depending on t). If $X'(t)$ is orthogonal to Z and $X(0)$ is orthogonal to Z, show that the path $X(t)$ is orthogonal to Z for all t.

   c) If the position $X(t)$ of a particle move on a sphere of radius R, so $\|X(t)\| = R$ show that the position and velocity are orthogonal for all t.

6. Say $X(t)$ is a solution of the differential equation $\dfrac{dX}{dt} = AX$, where A is an *anti-symmetric* matrix. Show that $\|X(t)\| = \text{constant}$.

7. Say $X(t)$ is a curve in $\mathbb{R}^3$ and $P \in \mathbb{R}^3$ is a point not on the curve. Assume there is a point $Q := X(t_0)$ on the curve that is closest to P. Show that the straight line from Q to P is perpendicular to the curve (that is, perpendicular to its tangent line at the point).

The next sequence of problems all concern the METHOD OF LEAST SQUARES. As a reference, see `http://www.math.upenn.edu/~kazdan/260S12/notes/vectors/vectors6.pdf` (Notes on inner products)

8. Use the Method of Least Squares to find the straight line $y = ax + b$ that best fits the following data given by the following four points (x_j, y_j), $j = 1, \ldots, 4$:

$$(-2, 4), \qquad (-1, 3), \qquad (0, 1), \qquad (2, 0).$$

   Ideally, you'd like to pick the coefficients a and b so that the four equations $ax_j + b = y_j$, $j = 1, \ldots, 4$ are all satisfied. Since this probably can't be done, one uses least squares to find the best possible a and b.

9. Find a curve of the form $y = a + bx + cx^2$ that best fits the following data

x	-2	-1	0	1	2	3	4
y	4	1.1	-0.5	1.0	4.3	8.1	17.5

10. The water level in the North Sea is mainly determined by the so-called M2 tide, whose period is about 12 hours. The height $H(t)$ thus roughly has the form

$$H(t) = c + a\sin(2\pi t/12) + b\cos(2\pi t/12), \tag{1}$$

   where time t is measured in hours (note $\sin(2\pi t/12$ and $\cos(2\pi t/12)$ are periodic with period 12 hours). Say one has the following measurements:

t (hours)	0	2	4	6	8	10
$H(t)$ (meters)	1.0	1.6	1.4	0.6	0.2	0.8

   Find the coefficients a, b, and c in (1) that best fit this data.

11. The comet Tentax, discovered only in 1968, moves within the solar system. The following are observations of its position (r, θ) in a polar coordinate system with center at the sun:

r	2.70	2.00	1.61	1.20	1.02
θ	48	67	83	108	126

(here θ is an angle measured in degrees).

By Kepler's first law the comet should move in a plane orbit whose shape is either an ellipse, hyperbola, or parabola (this assumes the gravitational influence of the planets is neglected). Thus the polar coordinates (r, θ) satisfy

$$r = \frac{p}{1 - e\cos\theta}$$

where p and the eccentricity e are parameters describing the orbit. Use the data to estimate p and e by the method of least squares. Hint: Make some (simple) preliminary manipulation so the parameters p and e appear *linearly*; then apply the method of least squares.

Bonus Problem

[Please give these directly to Professor Kazdan]

B-1 a) Compute $\max \int_{-1}^{1} x^3 h(x)\,dx$ where $h(x)$ is any continuous function on the interval $-1 \le x \le 1$ subject to the restrictions

$$\int_{-1}^{1} h(x)\,dx = \int_{-1}^{1} xh(x)\,dx = \int_{-1}^{1} x^2 h(x)\,dx = 0; \qquad \int_{-1}^{1} |h(x)|^2\,dx = 1.$$

b) Compute $\displaystyle \min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2\,dx.$

B-2 [Dual Variational Problems] Let $V \subset \mathbb{R}^n$ be a linear space, $Q : R^n \to V^\perp$ the orthogonal projection into $V^\perp$, and $x \in \mathbb{R}^n$ a given vector. Note that $Q = I - P$, where P in the orthogonal projection into V

a) Show that $\displaystyle \max_{\{z \perp V,\, \|z\|=1\}} \langle x,\, z\rangle = \|Qx\|.$

b) Show that $\displaystyle \min_{v \in V} \|x - v\| = \|Qx\|.$

[Remark: *dual variational problems* are a pair of maximum and minimum problems whose extremal values are equal.]

[Last revised: February 27, 2012]