Problem Set 2

DUE: In class Thursday, Jan. 26. *Late papers will be accepted until 1:00 PM Friday.*

1. Which of the following sets of vectors are bases for $\mathbb{R}^2$?

   a). $\{(0, 1),\ (1, 1)\}$
   b). $\{(1, 0),\ (0, 1),\ (1, 1)\}$
   c). $\{(1, 0),\ (-1, 0\}$
   d). $\{(1, 1),\ (1, -1)\}$
   e). $\{((1, 1),\ (2, 2)\}$
   f). $\{(1, 2)\}$

2. a) Compute the dimension of the intersection of the following two planes in $\mathbb{R}^3$

$$x + 2y - z = 0, \qquad 3x - 3y + z = 0.$$

   b) A map $L : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by the matrix $L := \begin{pmatrix} 1 & 1 & -1 \\ 3 & -3 & 1 \end{pmatrix}$. Find the nullspace (kernel) of L.

3. For which real numbers x do the vectors: $(\lambda, 1, 1, 1)$, $(1, \lambda, 1, 1)$, $(1, 1, \lambda, 1)$, $(1, 1, 1, \lambda)$ *not* form a basis of $\mathbb{R}^4$? For each of the values of x that you find, what is the dimension of the subspace of $\mathbb{R}^4$ that they span?

4. Compute the dimension and find bases for the following linear spaces.

   a) Real 4×4 matrices whose diagonal elements are zero.

   b) Quartic polynomials p with the property that $p(2) = 0$ and $p(3) = 0$.

   c) Cubic polynomials $p(x, y)$ in two real variables with the properties: $p(0, 0) = 0$, $p(1, 0) = 0$ and $p(0, 1) = 0$. [SUGGESTION: As a warm-up exercise, try quadratic polynomials $p(x, y) = a + bx + cy + dx^2 + exy + fy^2$ with $p(1, 0) = 0$.]

   d) The space of linear maps $L : \mathbb{R}^5 \to \mathbb{R}^3$ whose kernels contain $(0, 2, -3, 0, 1)$. [SUGGESTION: As a warm-up exercise, try linear maps $L : \mathbb{R}^3 \to \mathbb{R}^2$ whose kernels contain $(0, 2, -3)$.]

5. Let $C(\mathbb{R})$ be the linear space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

   a) Let S_c be the set of differentiable functions $u(x)$ that satisfy the differential equation

$$u' = 2xu + c$$

   for all real x. For which value(s) of the real constant c is this set a linear subspace of $C(\mathbb{R})$?

b) Let $C^2(\mathbb{R})$ be the linear space of all functions from $\mathbb{R}$ to $\mathbb{R}$ that have two continuous derivatives and let S_f be the set of solutions $u(x) \in C^2(\mathbb{R})$ of the differential equation

$$u'' + 2xu = f(x)$$

for all real x. For which polynomials $f(x)$ is the set S_f a linear subspace of $C(\mathbb{R})$? [Note: You are not being asked to solve this equation.]

c) Let $\mathcal{A}$ and $\mathcal{B}$ be linear spaces and $L : \mathcal{A} \to \mathcal{B}$ be a linear map. For which vectors $y \in \mathcal{B}$ is the set

$$\mathcal{S}_y := \{x \in \mathcal{A} \mid Lx = y\}$$

a linear space?

6. Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$.

a) Let $W := U \cap V$. Show that W is a linear space and find all possible values for the dimension of W.

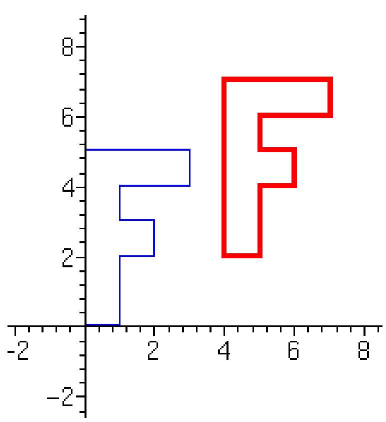
b) Let $Z := U + V$ as the set of all vectors $z = u + v$ where $u \in U$ and $v \in V$ can be any vectors. Show that W is a linear space and find all possible values for the dimension of W.

7. Linear maps $F(X) = AX$, where A is a matrix, have the property that $F(0) = A0 = 0$, so they necessarily leave the origin fixed. It is simple to extend this to include a translation,

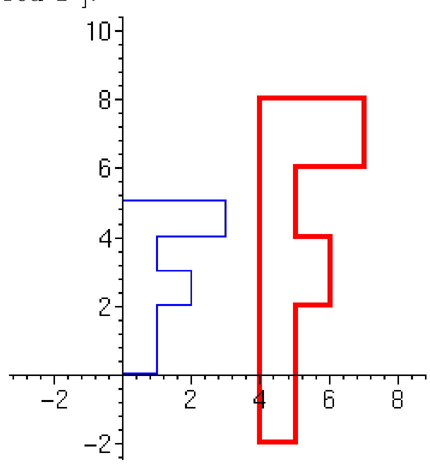
$$F(X) = V + AX,$$

where V is a vector. Note that $F(0) = V$. [These are called *affine* maps].

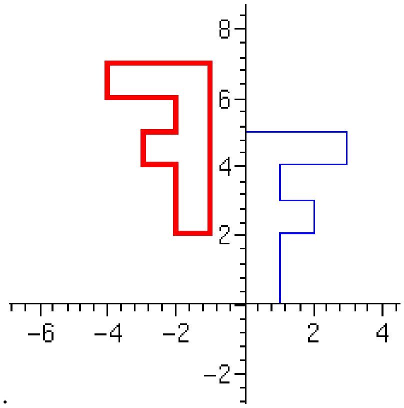
a) Find the vector V and the matrix A that describe each of the following mappings [here the light blue F is mapped to the dark red F].

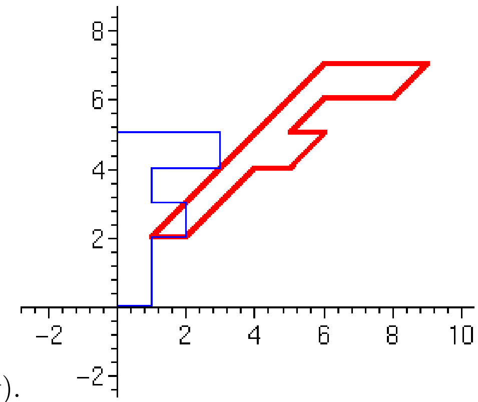


i).



ii).


iii).


iv).

b) Show that the set of *all* affine maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ forms a linear space. What is its dimension?

8. a) Find set Q of all linear maps $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ whose nullspace is exactly the plane $\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 0 \}$.

   b) Is this set a linear space? If so, what is its dimension?

Bonus Problem

[Please give these directly to Professor Kazdan]

1-B In the notes on Symmetries:

`http://www.math.upenn.edu/ kazdan/260S12/notes/symmetries.pdf`

do the Exercise at the bottom of page 2:

a) Use $RS = SR^3$ to show that the maps RSR, R^2S , and RSR^{-1} are in the list (5).

b) Prove that the list (5) really does contain all the symmetries of the square. I suggest beginning with the special case where the vertex A is fixed. What are the possible adjacent vertices? A key ingredient is that the symmetries of the square are rigid motions, that is, they preserve distances between points, so no stretching or shrinking is allowed.

[Last revised: January 24, 2012]