Math 260, Spring 2012 Jerry L. Kazdan

Problem Set 13

Due: Thursday April 26, 1 PM

Unless otherwise stated use the standard Euclidean *norm. Also all regions* $\Omega \subset \mathbb{R}^n$ *are assumed to be bounded, connected, and have smooth boundaries..*

Remark: The first two problems were originally on Exam 3, but at the last moment I deleted them fearing the exam was too long.

1. Let $\mathbf{V} = (y^2+1)\mathbf{i} + (2xy-4y)\mathbf{j} + 2\mathbf{k}$
   a) Find a function $u(x,y,z)$ so that $\mathbf{V} = \nabla u$.
   b) Let γ be the triangle bounded by the x-axis, the y-axis, and the straight line $2x+y=2$, traversed counterclockwise. Compute $\oint_\gamma \mathbf{V}\cdot d\mathbf{s}$.

2. a) Let $\Omega \subset \mathbb{R}^3$ be the region below the surface $z = 4-(x^2+y^2)$ and above the xy-plane. Compute $\iiint_\Omega z\,dV$.
   b) Let $\Omega \subset \mathbb{R}^3$ be the region below the surface $z = 4-(x^2+4y^2)$ and above the xy-plane. Compute $\iiint_\Omega z\,dV$.

3. Compute $\oint_\gamma x\,dy - y\,dx$ where the closed curve γ is the triangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(1,0)$, and $(1,2)$, traversed counterclockwise

4. [Marsden-Tromba, p. 437 # 6] Verify the Green's-Stokes' theorem in the plane $\oint_{\partial D} P\,dx + Q\,dy = \iint_D \dots$ for the region $[0,\,\frac{\pi}{2}]$, $[0,\,\frac{\pi}{2}]$, with $P(x,y) = \sin x$, $Q(x,y) = \cos y$. You should compute both sides of the formula to verify that they agree.

5. [Marsden-Tromba p. 437 # 11d]. Verify the Green's-Stokes' theorem in the plane or the disk D with center at the origin and radius R for $P(x,y) = 2y$, $Q(x,y) = x$.

6. [Marsden-Tromba p. 437 # 15]. Evaluate $\int_C (2x^3-y^3)\,dx + (x^3+y^3)\,dy$ where C is the unit circle both directly and using the Green's-Stokes' theorem in the plane.

7. [Marsden-Tromba p. 438 # 20]. Let $P(x,y) = -y/(x^2+y^2)$ and $Q(x,y) = x/(x^2+y^2)$ in the unit disc D. Show that Green's theorem fails for this P and Q. Explain why.

8. [Marsden-Tromba p. 439 # 38]. Use Green's theorem in the plane to prove the change of variables formuls in the following special case

$$\iint_D dx\, dy = \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

for a transformation $(u, v) \mapsto (x(u,v),\, y(u,v))$.

9. In applying the divergence theorem where the region is all of $\mathbb{R}^3$, the integral over the boundary is not well defined. Instead, one works on the ball of radius R and then lets $R \to \infty$.

Suppose $V(x,y,z)$ is a vector-valued function defined everywhere in 3-dimensional space. Further, suppose that V is differentiable and that for some constant c

$$\|V(x,y,z)\| \le \frac{c}{1 + (x^2+y^2+z^2)^{3/2}}$$

for all (x,y,z). Show that

$$\iiint_{\mathbb{R}^3} \nabla \cdot V(x,y,z)\, dx\, dy\, dz = 0. \tag{1}$$

In other words, if $B(0,R)$ is the ball of radius R centered at the origin, then (1) means that

$$\lim_{R\to\infty} \iiint_{B(0,R)} \nabla \cdot V(x,y,z)\, dx\, dy\, dz = 0.$$

10. a) Say $u(x)$ satisfies $u'' - c(x)u = 0$ on the boubded interval $a < x < b$ with $u(x) = 0$ on the boundary, so $u(a) = 0$ and $u(b) = 0$. Assuming that $c(x) \ge 0$, show that then the only possibility is $u(x) = 0$ throughout the interval. [SUGGESTION: Multiply the equation by u and integrate over the interval. Then integrate by parts.] The example $u'' + u = 0$ on $0 < x < \pi$, one of whose solutions is $\sin x$ shows that the assumption $c(x) \ge 0$ plays a vital role.

    b) Say $u(x,y)$ satisfies $\Delta u - c(x,y)u = 0$ in a bounded region Ω in the plane with $u(x,y) = 0$ on the boundary, $\partial\Omega$. Assuming that $c(x,y) \ge 0$, show that then the only possibility is $u(x,y) = 0$ throughout Ω.

    c) Let $u(x,y)$ and $v(x,y)$ satisfy $\Delta u - c(x,y)u = f(x,y)$ in Ω with $u(x,y) = \phi(x,y)$ on $\partial\Omega$, as well as $\Delta v - c(x,y)v = f(x,y)$ in Ω with $v(x,y) = \phi(x,y)$ on $\partial\Omega$, so they satisfy the same differential equation and the same boundary condition. As above, assume $c(x,y) \ge 0$. Show that $u = v$ throughout Ω.

11. a) [VIBRATING STRING] Let $u(x,t)$ be a solution of the wave equation $u_{tt} = u_{xx}$ in one space variable, say $0 \le x \le L$. Assume the ends of the string are fixed:

$u(0,t) = 0$ and $u(L,t) = 0$. Define the *energy* as

$$E(t) := \frac{1}{2}\int_0^L \left[u_t^2 + u_x^2\right]\,dx.$$

Show that *energy is conserved*: $dE/dt = 0$. [HINT: At some step of the computation integrate by parts using that because of the boundary condition, the velocity is zero at the end points.]

b) Use this to show that if the initial position and initial velocity are zero, so $u(x,y,0) = 0$, $u_t(x,y,0) = 0$, Then $(x,y,t) = 0$ for all $(x,y) \in \Omega$ and all $t \geq 0$.

c) [VIBRATING DRUMHEAD] Let $u(x,y,t)$ be a solution of the wave equation $u_{tt} = u_{xx} + u_{yy}$ for (x,y) in a bounded set Ω in $\mathbb{R}^2$ (the drumhead). Assume the drumhead is fixed along its boundary: $u(x,y,t) = 0$ for $(x,y) \in \partial\Omega$. Define the *energy* as

$$E(t) := \frac{1}{2}\iint_\Omega \left[u_t^2 + |\nabla u|^2\right]\,dx\,dy.$$

Show that *energy is conserved*: $dE/dt = 0$.

Bonus Problem

[Please give these directly to Professor Kazdan]

NOTATION: Let $u(x,y)$ be a smooth function on the plane (actually, we will only use that the second derivatives are continuous) and $D \subset \mathbb{R}^2$ be an open region. Given a point $p \in D$, let $B_r(p)$ be the closed disk of radius r centered at p and contained in D for $0 < r \leq R$ (so just pick R sufficiently small). Define $I(r)$ by

$$I(r) := \frac{1}{2\pi r}\int_{\partial B_r(p)} u\,ds.$$

This is just the average of u on this circle.

B-1 [Marsden-Tromba p. 438–9 # 29-34]

a) Show that $\lim_{r\to 0} I(r) = u(p)$.

b) Let $\mathbf{n}$ denote the unit outer normal to ∂B_r and define $\partial u/\partial n := \nabla u \cdot \mathbf{n}$ (this is the directional derivative of u in the direction of the outer normal). Show that

$$\int_{\partial B_r} \frac{\partial u}{\partial n}\,ds = \iint_{B_r} \Delta u\,dA.$$

c) Use this to show that $I'(r) = \dfrac{1}{2\pi r}\displaystyle\iint_{B_r} \Delta u\,dA$.

d) Suppose that u is a harmonic function, that is, $\Delta u = 0$ in D. Use the above to deduce the *mean value property of harmonic functions*

$$u(p) = \frac{1}{2\pi r} \int_{\partial B_r} u \, ds.$$

This states the the value of u at the center of a disk is the average of its values on the circumference.

e) From the previous part, deduce the "solid mean value property"

$$u(p) = \frac{1}{\pi R^2} \iint_{B_R} u \, dA.$$

f) If u is harmonic in D and has a local maximum at some point p in D, show that u must be a constant in some small disk centered at p.

g) Assuming that D is connected, show that if u is harmonic in D and has its *absolute maximum* at some point p in D (so $u(p) \geq u(q)$ for all points $q \in D$), then u must be a constant D.

Similarly, if u has its *absolute minimum* at some point p in D, then u must be a constant in D.

[Last revised: May 10, 2012]