Math 260, Spring 2012 Jerry L. Kazdan

Problem Set 12

DUE: never

Unless otherwise stated use the standard Euclidean *norm.*

1. Let Let $\gamma(t)$ be any smooth *closed* curve in $\mathbb{R}^4$. Why, with only a mental computation, is
$$\oint_\gamma 2x\,dx + 6(x-y)\,dy = \oint_\gamma 6x\,dy\ ?$$

2. Find some *closed* curve $\gamma(t)$ so that $\oint_\gamma 6x\,dy > 0$.

3. Let C be the portion of the unit circle $x^2+y^2=1$ with $x \ge 0$ oriented so that it begins at $(0,1)$ and ends at $(0,-1)$. Evaluate
$$\int_\gamma e^x \sin y\,dx + e^x \cos y\,dy.$$

4. Let $\mathbf{F}$ be a continuous force field defined on $\mathbb{R}^3$ and suppose that a particle of mass m moves along a path $X(t)$ determined by Newton's second law of motion, $mX'' = \mathbf{F}(X(t))$ during the time interval $a \le t \le b$. Show that
$$\int_a^b \mathbf{F}\cdot X'(t)\,dt = \frac{m}{2}\|X'(b)\|^2 - \frac{m}{2}\|X'(a)\|^2.$$
In physics, the right hand side is interpreted as a change in *kinetic energy*.

5. Let $\psi(t)$ be a scalar-valued function with a continuous derivative for $0 < t < \infty$ and let $\mathbf{X} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \in \mathbb{R}^3$. Define the vector field $\mathbf{F}(x,y,z) := \psi(\|\mathbf{X}\|)\mathbf{X}$ for all $\mathbf{X} \ne 0$. Show that this vector field is conservative by finding a scalar-valued function $\varphi(r)$ with the property that $\mathbf{F}(\mathbf{X}) := \nabla\varphi(\|\mathbf{X}\|)$. In particular, this shows that *every central force field is conservative* except possibly at the origin.

6. [Marsden-Tromba p. 381 #2] A surface in $\mathbb{R}^3$ is defined by $x = u^2 - v^2$, $y = u + v$, $z = u^2 + 4v$.
   a) At what points is this surface *regular*?
   b) Find the equation of the tangent plane at $(\frac{-1}{4}, \frac{1}{2}, 2)$ (so $u = 0$, $v = \frac{1}{2}$).

7. [Marsden-Tromba p. 381 #7] Match the parametrization as belonging to the surfaces: (i) ellipsoid, (ii) parabolic cylinder, (iii) hyperboloid, or (iv) cone [Corresponding drawings are in the text].

a) $\Phi(u,v) := ((2\sqrt{1+u^2})\cos v,\ (2\sqrt{1+u^2})\sin v,\ u)$

b) $\Phi(u,v) := (3\cos u \sin v,\ 2\sin u \sin v, \cos v)$

c) $\Phi(u,v) := (u,\ v,\ u^2)$

d) $\Phi(u,v) := (u\cos v,\ u\sin v,\ u)$

8. [Marsden-Tromba p. 383 #18] For a sphere in $\mathbb{R}^3$ centered at the origin with radius 2, find the equation of the tangent plane at the point $(1,1,1/\sqrt{2})$ by considering the sphere as

   a) a surface parametrized by $\Phi(\theta,\phi) := (2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi)$,

   b) a level surface of $f(x,y,z) := x^2+y^2+z^2$,

   c) the graph of $g(x,y) := \sqrt{4-x^2-y^2}$.

9. We used the following parametrization of the torus:

$$x(\theta,\phi) = (3+\cos\phi)\cos\theta, \qquad y(\theta,\phi) := (3+\cos\phi)\sin\theta, \qquad z(\theta,\phi) := \sin\phi,$$

   where $0 \le \theta,\ \phi \le 2\pi$. Show that the image surface (our torus) is *regular* at all points.

10. Compute the area of the torus using the parametrization of the above problem.

11. [Marsden-Tromba p. 392 #26d] Consider the graph of $z := y^3\cos^2 x$ over the triangle with vertices at $(-1,1)$, $(0,2)$, $(1,1)$. Express the surface area as a double integral (but don't evaluate it).

12. [Marsden-Tromba p. 398 #4] Evaluate the integral $\iint_S (x+z)\,dS$, where S is the part of the cylinder $y^2+z^2=4$ with $0 \le x \le 5$.

13. [Similar to Marsden-Tromba p. 399 #23] Let S be a two dimensional surface in $\mathbb{R}^n$, $n \ge 3$, given by the parametrization $(u,v) \mapsto \Phi(u,v)$ with

$$x_1 = x_1(u,v), \quad x_2 = x_2(u,v), \quad \ldots \quad x_n = x_n(u,v).$$

   Say we have a curve $\gamma(t) = (u(t),\ v(t))$. Then its image under Φ gives a curve $X(t) := \Phi(u(t),\ v(t))$ in the surface in $\mathbb{R}^n$. As usual, the element of arc length is given by $ds = \|\gamma'(t)\|dt$. Show that

$$\left(\frac{ds}{dt}\right)^2 = E(u,v)\left(\frac{du}{dt}\right)^2 + 2F(u,v)\frac{du}{dt}\frac{dv}{dt} + G(u,v)\left(\frac{dv}{dt}\right)^2, \tag{1}$$

where

$$E(u,v) = \left\| \frac{\partial \Phi}{\partial u} \right\|^2, \qquad F(u,v) = \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v}, \qquad G(u,v) = \left\| \frac{\partial \Phi}{\partial v} \right\|^2.$$

We think of the formula (1) as defining an inner product on tangent vectors and use the symmetric matrix

$$g := \begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix}.$$

Since the element of arc length is always positive on non-trivial curves, this matrix g is *required* to be positive definite. The element of area on the surface is $dS := \sqrt{\det g}\, du\, dv = \sqrt{EG - F^2}\, du\, dv$. Note that this works in any dimension (the cross product version works only in dimension $n = 3$).

We often write equation (1) as

$$ds^2 = E(u,v)\, du^2 + 2F(u,v)\, du\, dv + G(u,v)\, dv^2$$

and refer to it as specifying a *Riemannian Metric* on the surface.

[Last revised: April 11, 2012]