Math 260, Spring 2012 Jerry L. Kazdan

Problem Set 11

DUE: Thurs. April 5. *Late papers will be accepted until 1:00 PM Friday.*

1. Let $u(x,t) := \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$. Show that u satisfies the wave equation (for a vibrating string) $u_{tt} = c^2 u_{xx}$ with initial position $u(x,0) = f(x)$ and initial velocity $u_t(x,0) = g(x)$.

   [SUGGESTION: If $G(p,q) := \int_p^q h(s)\,ds$, compute $\partial G(p,q)/\partial p$ and $\partial G(p,q)/\partial q$].

2. Let $Q \subset \mathbb{R}^3$ be the portion of the shell $1 \le x^2 + y^2 + z^2 \le 9$ that is in the first octant $x \ge 0,\ y \ge 0,\ z \ge 0$.
   a) Set-up and evaluate the triple integral to compute the volume of Q.
   b) Compute $\iiint_Q z\,dV$.

3. [Marsden-Tromba, p.337 #2] Assuming uniform density, find the coordinates of the center of mass of the semicircle $y = \sqrt{r^2 - x^2} \ge 0$.

4. [Marsden-Tromba, p.337 #4] Find the average of e^{x+y} over the triangle with vertices at $(0,0)$, $(1,0)$, and $(0,1)$.

5. [Marsden-Tromba, p.338 #16] Find the average of e^{-z} over the unit ball $x^2+y^2+z^2 \le 1$.

6. [Marsden-Tromba, p.338 #24–25]
   a) Let D be a region in the part of the xy-plane with $x > 0$. Assume D has uniform density. Let $A(D)$ be its area and $(\bar{x}, \bar{y})$ its center of mass. Let W be the solid obtained by rotating D about the y-axis. Show that $\operatorname{Vol}(W) = 2\pi\bar{x}A(D)$. [Note that $2\pi\bar{x}$ is the length of the circle traversed by the center of mass.]
   b) Apply this to compute the volume of the torus obtained by rotating the unit circle centered at $(3,0)$ around the y-axis.

7. [Marsden-Tromba, P. 373 #3] Let $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Evaluate the line integral of $\mathbf{F}$ along each of the following paths:

   a). $\mathbf{c}(t) = (t,t,t), \quad 0 \le t \le 1$
   b). $\mathbf{c}(t) = (\cos t, \sin t, 0), \quad 0 \le t \le 2\pi$
   c). $\mathbf{c}(t) = (\sin t, 0, \cos t), \quad 0 \le t \le 2\pi$
   d). $\mathbf{c}(t) = (t^2, 3t, 2t^3), \quad -1 \le t \le 2$

8. Repeat the previous problem with $\mathbf{F}(x,y,z) = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$.

9. [Marsden-Tromba, P.374 #13] Let $\mathbf{c}(t)$ be a path and $\mathbf{T}$ the unit tangent vector. What is $\int_{\mathbf{c}} \mathbf{T} \cdot d\mathbf{s}$?

10. Let $\mathbf{r} := x\mathbf{i} + y\mathbf{j}$ and $\mathbf{V}(x,y) := p(x,y)\mathbf{i} + q(x,y)\mathbf{j}$ be (smooth) vector fields and C a smooth curve in the plane. In this problem J is the line integral $J = \int_C \mathbf{V} \cdot d\mathbf{r}$. For each of the following, either give a proof or give a counterexample.

    a) If C is a vertical line segment and $q(x,y) = 0$, then $J = 0$.

    b) If C is a circle and $q(x,y) = 0$, then $J = 0$.

    c) If C is a circle centered at the origin and $p(x,y) = -q(x,y)$, then $J = 0$.

    d) If $p(x,y) > 0$ and $q(x,y) > 0$, then $J > 0$.

11. Let $\Omega \subset \mathbb{R}^3$ be the half-ball where $x^2 + y^2 + z^2 \le 1$ and $z \ge 0$. Use symmetry to deduce (without compuation) that $\iiint_\Omega x^3 \, dV = 0$.

12. If $\mathbf{F} := F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a *conservative field*, then $\mathbf{F} = \nabla u(x,y,z)$ for some scalar-valued function u and u is called the *potential function* for the field $\mathbf{F}$.

    a) If $\mathbf{F}$ is conservative, show that

    $$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \text{and} \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

    b) Show that $\mathbf{F} := 2x\mathbf{i} + z\mathbf{j} + 2y\mathbf{k}$ is *not* conservative.

    c) Show that $\mathbf{F} := 2xy^3\mathbf{i} + 3x^2y^2\mathbf{j}$ is conservative by finding the potential function $u(x,z)$.

    d) Show that $\mathbf{F} := 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ is conservative by finding the potential function u.

13. [Marsden-Tromba, P.374 #17] Evaluate $\displaystyle\int_C 2xyz\,dx + x^2z\,dy + x^2z\,dz$ where C is an oriented simple curve connecting $(1,1,1)$ to $(1,2,4)$.

14. [Marsden-Tromba, P.374 #18] Suppose $\nabla f(x,y,z) = 2xyze^{x^2}\mathbf{i} + ze^{x^2}\mathbf{j} + ye^{x^2}\mathbf{k}$. If $f(0,0,0) = 5$, find $f(1,1,2)$.

Bonus Problems

[Please give this directly to Professor Kazdan]

B-1 Let $V_n(R)$ be the "volume" of the ball $B_n(R) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \cdots x_n^2 \le R^2\}$ and $A_{n-1}(R)$ the "area" of the sphere S^{n-1} of radius R in $\mathbb{R}^n$, so $x_1^2 + \cdots x_n^2 = R^2$.

Complete the outline begun in class to obtain the recursion formula

$$A_n(1) = \frac{2\pi}{n-1} A_{n-2}(1)$$

and use this to find formulas for $V_n(R)$ and $A_{n-1}(R)$. There are two cases depending if n is even or odd.

Show that $\lim_{n\to\infty} V_n(1) = 0$.

B-2 Let $\Omega \subset \mathbb{R}^2$ be a connected open set. If $u(x, y)$ is a smooth scalar-valued function with the property that $\nabla u = 0$ throughout Ω, show that $u(x, y)$ must be a constant.

[Last revised: April 1, 2012]