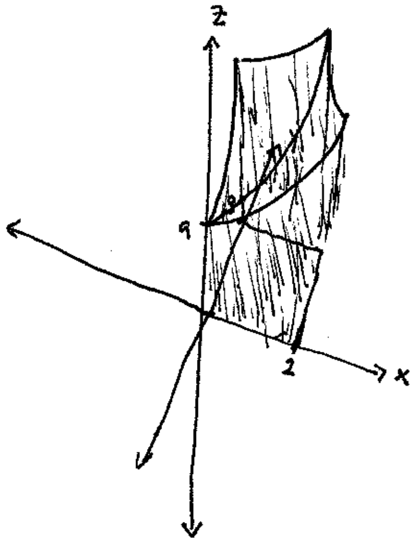
1. Sketch the solid whose volume is given by

$$\int_0^3 \left(\int_0^2 (9 + x^2 + y^2)\, dx \right) dy .$$

—



Sketch $z = 9 + x^2 + y^2$ from $0 \le x \le 2$, $0 \le y \le 3$, and the area under its curve is our solid.

Volume: $$\int_0^3 \left[9x + \frac{x^3}{3} + xy^2 \right]_0^2 dy$$

$$\parallel$$

$$\int_0^3 18 + \frac{8}{3} + 2y^2 \; dy$$

$$\parallel$$

$$\left[\left(18 + \frac{8}{3} \right) y + \frac{2y^3}{3} \right]_0^3 = 54 + 8 + 18 = \underline{80}.$$

2. Let $f(x,y)$ be continuous on $[a,b] \times [c,d]$. For $a \le x \le b$, $c \le y \le d$ define

$$G(x,y) := \int_a^x \left(\int_c^y f(u,v)\, dv \right) du$$

Show that $\dfrac{\partial^2 G}{\partial x \partial y} = \dfrac{\partial^2 G}{\partial y \partial x} = f(x,y)$.

—

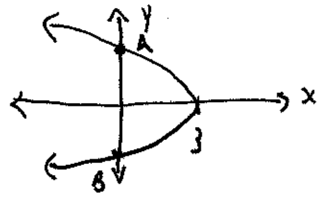
Proof: Using the fact that $G(x,y)$ will be a C^2 function, we can assert that $\dfrac{\partial^2 G}{\partial x \partial y} = \dfrac{\partial^2 G}{\partial y \partial x}$, but since we never particularly pay attention to the details of interchanging mixed partials, don't worry if you didn't point this out. It's easy enough to check this by hand, so we'll proceed:

Define $F(u,y) := \displaystyle\int_c^y f(u,v)\, dv$. Now $G(x,y) = \displaystyle\int_a^x F(u,y)\, du$ and we can use the single-variable Fundamental Theorem of Calculus:

$$\frac{\partial G(x,y)}{\partial x} = \frac{\partial}{\partial x} \int_a^x F(u,y)\, du = F(x,y) \quad \text{and} \quad \frac{\partial^2 G(x,y)}{\partial y\, \partial x} = \frac{\partial}{\partial y} F(x,y) = \frac{\partial}{\partial y} \int_c^y f(x,v)\, dv = f(x,y).$$

We can interchange integrals: $G(x,y) = \displaystyle\int_c^y \int_a^x f(u,v)\, du\, dv$, as $[a,x] \times [c,y]$ is a rectangle in the uv-plane. Either compute $\dfrac{\partial^2 G}{\partial x \partial y}$ this way, or use the assertion tomorrow.

3. D: bounded by $\cdot$ y-axis
$\cdot\ x = -4y^2+3$.



Compute $\iint_D x^3 y \, dx\, dy \doteq I$

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x goes from $x=0$ to $x=-4y^2+3$, y goes from A to B.

Compute : $0=-4y^2+3$, $y^2 = 3/4$, $y = \pm \frac{\sqrt{3}}{2}$. $A = \sqrt{3}/2$, $B = -\sqrt{3}/2$.

So $I = \int_{-\frac{\sqrt{3}}{2}}^{\sqrt{3}/2}\int_0^{3-4y^2} x^3 y \, dx\, dy = \int_{-\sqrt{3}/2}^{\sqrt{3}/2}\left[\frac{x^4 y}{4}\right]_0^{3-4y^2} dy = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{(3-4y^2)^4}{4}\, y \, dy.$

Let $u = 3-4y^2$, $du = -8y\, dy$. If we convert our limits now, we'll see our integral goes from $u=0$ to $u=0$ and $I=0$, but we'll postpone the obvious:

$$I = \int_{y=-\sqrt{3}/2}^{y=\sqrt{3}/2} \frac{u^4}{4}\, y \, \frac{du}{-8y} = -\int_{y=-\sqrt{3}/2}^{y=\sqrt{3}/2} \frac{u^4}{32}\, du = \left[\frac{-u^5}{160}\right]_{y=-\sqrt{3}/2}^{y=\sqrt{3}/2} = \cancel{\frac{-\sqrt{3}^{5/2}}{32\cdot 160} + \frac{(-\sqrt{3})^{5/2}}{32\cdot 160}}$$

$u \neq y!$ People did this.

$$= \left[\frac{-(3-4y^2)^5}{160}\right]_{-\sqrt{3}/2}^{\sqrt{3}/2} = 0 - 0 = 0$$ and we can hide no longer in the name of practicing technical ability.

Admittedly, we can notice symmetry over the x-axis and that our function is odd in x, and so our integral was always fated to be $= 0$.

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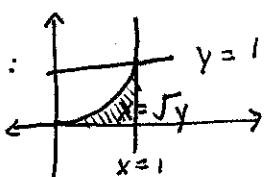
4. Change the order of integration and evaluate $\int_0^1\int_{\sqrt{y}}^1 e^{x^3}\, dx\, dy = I$

Region:



so ~~so~~ so $0 \le y \le x^2$,
$0 \le x \le 1$. Now we can make progress!

$$I = \int_0^1\int_0^{x^2} e^{x^3}\, dy\, dx = \int_0^1 \left[y e^{x^3}\right]_{y=0}^{y=x^2} dx = \int_0^1 x^2 e^{x^3}\, dx. \quad u = x^3,\ du = 3x^2\, dx \quad 0^3 = 0,\ 1^3 = 1.$$

$$I = \int_{u=0}^{u=1} \frac{e^u}{3}\, du = \left[\frac{e^u}{3}\right]_{u=0}^{u=1} = \boxed{\frac{e}{3} - \frac{1}{3}.}$$

Keep this idea in mind - also, get accustomed to using geometric pictures to keep track of changing orders of integration.

5. $\iiint_W f\, dW = \int_a^b \left(\int_?^? \left(\int_?^? f(x,y,z)\, dz \right) dy \right) dx$, $W = \sqrt{x^2+y^2} \le z \le 1$.

z, by our given description goes from $\sqrt{x^2+y^2}$ to 1.

Now we need to write bounds for y, then x, given that at most, $\sqrt{x^2+y^2} \le 1$. So $y^2 \le 1-x^2$: $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$.

Finally, $-1 \le x \le 1$ as $\sqrt{1-x^2} \ge 0$ and is well-defined over $\mathbb{R}$ by geometric assumption.

So we have $\iiint_W f\, dy = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\, dz\, dy\, dx.$

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6. W: $x=0$, $y=0$, $z=0$, $x+y=1$, $z=x+y$.

Evaluate

$$I = \iiint_W x\, dx\, dy\, dz.$$

As W is bounded in $\mathbb{R}^3$, and $x \ge 0$ on W, x is continuous on $\mathbb{R}^3$, our integral is finite and by Fubini's Thm,

$$I = \iiint_W x\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \int_0^{x+y} x\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \Big[xz \Big]_{0=z}^{x+y=z} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} x^2 + yx\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^2 x}{2} \right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 x^2(1-x) + \frac{(1-x)^2 x}{2}\, dx = \int_0^1 x^2 - x^3 + \frac{x - 2x^2 + x^3}{2}\, dx$$

$$= \int_0^1 -\frac{x^3}{2} + \frac{x}{2}\, dx = \left[-\frac{x^4}{8} + \frac{x^2}{4} \right]_0^1 = \frac{1}{4} - \frac{1}{8} = \boxed{\frac{1}{8}}$$

7. a) If A is a $n\times n$ matrix and $c \in \mathbb{R}$, show that $\det(cA) = c^n \det(A)$.

Pf: We use that if $\{v_1, \dots, v_n\}$ are our rows of A, then

$$\det\{cv_1, \dots, v_n\} = c \det\{v_1, \dots, v_n\} \quad \forall c \in \mathbb{R} \text{ for any row in } A.$$

Applied n times:

$$\det(cA) = \det\{cv_1, \dots, cv_n\} = c^n \det\{v_1, \dots, v_n\} = c^n \det(A).$$

b) If A and B are similar, show that $\det A = \det B$.

Pf: If A and B are similar, $\exists$ invertible matrix S such that $A = SBS^{-1}$.

We use that $\det(AB) = \det(A)\det(B)$:

$$A = SBS^{-1}$$
$$\det(A) = \det(SBS^{-1})$$
$$\det(A) = \det(S)\det(B)\det(S^{-1})$$
$$\det(A) = \det(B)\cdot\det(S)\cdot\det(S^{-1})$$
$$\det(A) = \det(B)\cdot\det(SS^{-1})$$
$$\det(A) = \det(B)\cdot\det(I) \qquad [\det(I) = 1.]$$
$$\det(A) = \det(B).$$

8. compute $\det A$, $A = \begin{bmatrix} a & 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 0 \\ c & 0 & 0 & 1 & -b \\ c & 0 & 0 & 1 & -a \\ d & e & 1 & f & g \end{bmatrix}$

$$\det A = -\det \begin{bmatrix} 0 & 1 & a & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & -b \\ 0 & 0 & c & 1 & -a \\ 1 & e & d & f & g \end{bmatrix} = \det \begin{bmatrix} 1 & e & d & f & g \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & -b \\ 0 & 0 & c & 1 & -a \\ 0 & 1 & a & 0 & 0 \end{bmatrix} \cdot (-1)^2$$

$$= \det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & -b \\ 0 & 0 & c & 1 & -a \\ 0 & 1 & a & 0 & 0 \end{bmatrix} + \underbrace{\det \begin{bmatrix} 0 & e & d & f & g \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & -b \\ 0 & 0 & c & 1 & -a \\ 0 & 1 & a & 0 & 0 \end{bmatrix}}_{=0}$$

8, continued

$$\det A = \det\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & 0 & 1 & -b \\ 0 & 0 & c & 1 & -a \\ 0 & 1 & a & 0 & 0 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & 0 & 1 & -b \\ 0 & 0 & c & 1 & -a \\ 0 & 0 & a-b & 0 & 0 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -b \\ 0 & 0 & c & 1 & -a \\ 0 & 0 & a-b & 0 & 0 \end{bmatrix} + \underbrace{\det\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 1 & -b \\ 0 & 0 & c & 1 & -a \\ 0 & 0 & a-b & 0 & 0 \end{bmatrix}}_{= 0}$$

$$= \det\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a-b & 0 & 0 \\ 0 & 0 & c & 1 & -a \\ 0 & 0 & 0 & 1 & -b \end{bmatrix} \cdot (-1) = -\det\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a-b & 0 & 0 \\ 0 & 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 & -b \end{bmatrix} - \underbrace{\det\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 1 & -a \\ 0 & 0 & 0 & 1 & -b \end{bmatrix}}_{= 0}$$

$$= -\det\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a-b & 0 & 0 \\ 0 & 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 0 & -b+a \end{bmatrix} =$$

$$\underbrace{-\det\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a-b & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & +(a-b) \end{bmatrix}}_{= -(a-b)^2 \det I} - \underbrace{\det\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a-b & 0 & 0 \\ 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & -(a-b) \end{bmatrix}}_{= 0}$$

$$= -(a-b)^2.$$

9. Case 1: one of the diagonals is zero (i.e. one of $a_{11}, a_{22}, \dots, a_{nn}$). Then we'll have linear dependency of our n rows:

$$A = \begin{bmatrix} a_{11} & \times & \times & \cdots & \times \\ & a_{22} & \times & \cdots & \times \\ & & \ddots & & \vdots \\ & 0 & & \ddots & \\ & & & & a_{nn} \end{bmatrix}$$

If a_{kk} is zero, we'll have a ~~Ek vector~~ k-dimensional subspace with only $k-1$ vectors left to span it, so $\det A = 0$.

so $\det A = 0 = \underbrace{0 \cdot \ldots \ldots}_{\text{diagonal entries}}$.

Now use induction: $n = 1$: $\det A = a_{11}$. Hooray!

$n = k-1$ true, assume A is $k \times k$.

$$\det A = \begin{bmatrix} a_{11} & a_{12} & \times & \cdots & a_{1k} \\ & a_{22} & & & \vdots \\ & & \ddots & & \vdots \\ & 0 & & \ddots & \vdots \\ & & & & a_{kk} \end{bmatrix} \; ?$$

Let $\{a_{11}, a_{12}, \dots, a_{1k}\} = v + v^*$,

$v = \{a_{11}, 0, \dots, 0\}$,

$v^* = \{0, a_{12}, \dots, a_{1k}\}$

Split the row up into v and v^*, use linearity:

$$\det A = \begin{bmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ 0 & a_{22} & \times & \cdots & \times \\ \vdots & & \ddots & & \vdots \\ \vdots & & 0 & \ddots & \times \\ \cdot & & & & a_{kk} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \cdots & \cdots & a_{1k} \\ 0 & a_{22} & \times & \cdots & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & & & & a_{kk} \end{bmatrix}$$

$\parallel$ 0 (column of 0's)

and this determinant is $a_{11} \cdot$ determinant of a $(k-1) \times (k-1)$ upper triangular matrix. By induction, we have our result.

10. $I = \iint_\Omega (x+y)^2 e^{x-y} \, dx \, dy$, Ω: $x+y = 1$, $x+y=4$, $x-y=-1$ and $x-y=1$.

Let $u = x+y$, $v = x-y$.

Our Jacobian's: $\left| \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix} \right| = \left| \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right| = |-2| = 2$ so $du\,dv = 2\,dx\,dy$

so $I = \dfrac{1}{2} \displaystyle\int_{-1}^{1}\int_{1}^{4} u^2 e^v \, du \, dv = \frac{1}{2}\int_{1}^{4} u^2 \, du \int_{-1}^{1} e^v \, dv$ by Fubini's:

$$= \left[\frac{u^3}{3}\right]_{1}^{4} \cdot \left[e^v\right]_{-1}^{1} \left(\frac{1}{2}\right) = \cancel{\frac{2}{3}(e^4 - e)} = \boxed{\frac{21}{2}(e - 1/e)}$$

11. Integrate $x^2+y^2+z^2$ over the cylinder $x^2+y^2 \le 2$, $-2 \le z \le 3$.

~~let~~ ~~...~~ ~~$x\cos\theta = r$~~

Okay, let $r\cos\theta = x$, $r\sin\theta = y$, $z = z$ so our Jacobian $\left(\frac{(r,\theta,z)}{(x,y,z)}\right)$

$$= \left|\begin{bmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} & \frac{dx}{dz} \\ dy/dr & dy/d\theta & dy/dz \\ dz/dr & dz/d\theta & dz/dz \end{bmatrix}\right|^{-1} = \left|\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ (r\sin\theta) & (r\cos\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}\right|^{-1} = \frac{1}{|r|},$$

$dr\, d\theta\, dz = \frac{1}{r}\, dx\, dy\, dz$

$\Rightarrow dx\, dy\, dz = r\, dr\, d\theta\, dz$

so our integral is

$$\int_{-2}^{3}\int_{0}^{2\pi}\int_{0}^{\sqrt{2}} (r^2+z^2)\, r \; dr\, d\theta\, dz$$

$$= \int_{-2}^{3}\int_{0}^{2\pi} \left[\frac{r^4}{4} + \frac{r^2 z^2}{2}\right]_0^{\sqrt{2}} d\theta\, dz$$

$$= \int_{-2}^{3} 2\pi\left(1+z^2\right) dz = 2\pi\left[z + \frac{z^3}{3}\right]_{-2}^{3}$$

$$= 2\pi(3+9) - 2\pi(-2-8/3)$$

$$= 2\pi(12 + 14/3)$$

$$= \frac{100\pi}{3}$$

12) a) $$\iint_{\mathbb{R}^2} \frac{dx\, dy}{[1+x^2+y^2]^2}$$ go polar:

$$\int_0^{2\pi}\int_0^{\infty} \frac{r\, dr\, d\theta}{(1+r^2)^2}$$ U-sub: $u = 1+r^2$, $du = 2r\,dr$; $r=0 \Rightarrow u=1$, $r \to \infty \Rightarrow u \to \infty$.

$$= \int_0^{2\pi}\int_1^{\infty} \frac{du}{2u^2}\, d\theta = \frac{2\pi}{2}\left[\int_1^{\infty}\frac{du}{u^2}\right] = \pi\left[\lim_{a\to\infty}\frac{-1}{a} - \frac{-1}{1}\right] =$$

$$\pi\left[0+1\right] = \pi.$$

b) $$\iint_{\mathbb{R}^2} \frac{dx\, dy}{(1+x^2+4y^2)^2}$$ $u = x$, $v = 2y$, $du\, dv = \left|\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right| dx\, dy$

$dx\, dy = \frac{du\, dv}{2}$

$$= \iint_{\text{still } \mathbb{R}^2} \frac{du\, dv}{(1+u^2+v^2)^2}\,\frac{1}{2} = \frac{\pi}{2} \text{ from a)}$$

c) $$\iint_{\mathbb{R}^2} \frac{dx\, dy}{[1+x^2+2xy+5y^2]^2}$$ $u = x+y$, $v = 2y$

$1+u^2+v^2 = 1+x^2+2xy+5y^2$

$du\, dv = \left|\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\right| dx\, dy$

$dx\, dy = \frac{du\, dv}{2}$

$$= \iint_{\text{still } \mathbb{R}^2} \frac{du\, dv}{(1+u^2+v^2)^2}\,\frac{1}{2} = \frac{\pi}{2} \text{ from a)}$$

d) Let A be a positive definite 2×2 symmetric real matrix and $X := (x_1, x_2) \in \mathbb{R}^2$. Show that

$$\iint_{\mathbb{R}^2} \frac{dx_1\, dx_2}{[1+\langle X, AX\rangle]^2} = \frac{\pi}{\sqrt{\det A}}.$$

(We can do this marvelously quickly using the Spectral Theorem, but we will proceed concretely.)

Let $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$

$a, b, c \in \mathbb{R}$, $a > 0$, $ab - c^2 > 0$ by hypothesis.

$$\langle X, AX\rangle = [x_1 \; x_2]\begin{bmatrix} a & c \\ c & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax_1^2 + 2cx_1x_2 + bx_2^2.$$

We complete the square:

$$\underline{ax_1^2 + 2cx_1x_2 + \frac{c^2}{a}x_2^2 - \frac{c^2}{a}x_2^2} + bx_2^2 \qquad \text{(okay as } a>0)$$

$$\Rightarrow \; = \left(\sqrt{a}\,x_1 + \frac{c}{\sqrt{a}}x_2\right)^2 + \left(\sqrt{b-\frac{c^2}{a}}\right)^2 x_2^2 \qquad \text{okay as } a>0,\; ab-c^2>0 \Rightarrow b-\frac{c^2}{a}>0$$

Now let $u_1 = \sqrt{a}\,x_1 + \frac{c}{\sqrt{a}}x_2$, $v = \sqrt{b-\frac{c^2}{a}}\;x_2$

Jacobian:
$$\begin{vmatrix} \sqrt{a} & c/\sqrt{a} \\ 0 & \sqrt{b-c^2/a} \end{vmatrix} = \sqrt{a}\sqrt{b-c^2/a} = \sqrt{ab-c^2}$$

so $du\,dv = \sqrt{ab-c^2}\,dx\,dy$ but $\sqrt{ab-c^2} = \sqrt{\det A}$

$$\Rightarrow \iint_{\mathbb{R}^2} \frac{du\,dv}{(1+u^2+v^2)^2}\frac{1}{\sqrt{\det A}} = \frac{\pi}{\sqrt{\det A}} \quad \text{from \#1.}$$

13) Compute $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy$ $\quad$ $a=b=1$

$x^2+y^2 = r^2$, $r\cos\theta = x$, $r\sin\theta = y$, $\quad r\,dr\,d\theta = dx\,dy$

$$\int_0^{2\pi}\int_0^{\infty} re^{-r^2}\,dr\,d\theta = 2\pi\int_0^{\infty} re^{-r^2}\,dr \qquad \begin{array}{l} u=r^2 \\ du = 2r\,dr \\ 0\to 0,\; \infty\to\infty \end{array}$$

$$= \pi\int_0^{\infty} e^{-u}\,du = \pi$$

now
$$\iint_{\mathbb{R}^2} e^{-(ax^2+by^2)}\,dx\,dy = \iint_{\mathbb{R}^2} e^{-(\sqrt{a}^2x^2 + \sqrt{b}^2y^2)} \qquad \text{okay as } a>0,\; b>0$$

$u = \sqrt{a}\,x$, $v = \sqrt{b}\,y$

$du\,dv = \sqrt{ab}\,dx\,dy$,

$$= \iint_{\mathbb{R}^2} e^{-(u^2+v^2)}\frac{du\,dv}{\sqrt{ab}} = \boxed{\frac{\pi}{\sqrt{ab}}} \quad \text{from above.}$$