Math 260, Spring 2012 Jerry L. Kazdan

Problem Set 1

DUE: In class Thursday, Jan. 19. *Late papers will be accepted until 1:00 PM Friday.*

These problems are intended to be straightforward with not much computation.

1. At noon the minute and hour hands of a clock coincide.
   a) What in the first time, T_1, when they are perpendicular?
   b) What is the next time, T_2, when they again coincide?

2. Which of the following sets are linear spaces?
   a) $\{X = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with the property $x_1 - 2x_3 = 0\}$
   b) The set of solutions x of $Ax = 0$, where A is an $m \times n$ matrix.
   c) The set of 2×2 matrices A with $\det(A) = 0$.
   d) The set of polynomials $p(x)$ with $\int_{-1}^{1} p(x)\,dx = 0$.
   e) The set of solutions $y = y(t)$ of $y'' + 4y' + y = 0$. [NOTE: You are *not* being asked to solve this differential equation. You are only being asked a more primitive question.]

3. Consider the system of equations

$$\begin{aligned} x + y - \ z &= a \\ x - y + 2z &= b \\ 3x + y \qquad &= c \end{aligned}$$

   a) Find the general solution of the homogeneous equation.
   b) If $a = 1$, $b = 2$, and $c = 4$, then a particular solution of the inhomogeneous equations is $x = 1$, $y = 1$, $z = 1$. Find the most general solution of these inhomogeneous equations.
   c) If $a = 1$, $b = 2$, and $c = 3$, show these equations have *no* solution.
   d) If $a = 0, b = 0, c = 1$, show the equations have *no* solution. [Note: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$].
   e) Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}$. Compute $\det A$.

   [Remark: After you have done parts a), and c), it is possible immediately to write the solutions to the remaining parts with no additional computation.]

4. a) Find a real 2×2 matrix A (other than $A = I$) such that $A^2 = I$.

b) Find a real 2×2 matrix A (other than $A = I$) such that $A^3 = I$.

5. a) Find a 2×2 matrix that rotates the plane by $+45$ degrees ($+45$ degrees means 45 degrees *counterclockwise*).

   b) Find a 2×2 matrix that rotates the plane by $+45$ degrees followed by a reflection across the horizontal axis.

   c) Find a 2×2 matrix that reflects across the horizontal axis followed by a rotation the plane by $+45$ degrees.

   d) Find a matrix that rotates the plane through $+60$ degrees, keeping the origin fixed.

   e) Find the inverse of each of these maps.

6. a) Find a 3×3 matrix that acts on $\mathbb{R}^3$ as follows: it keeps the x_1 axis fixed but rotates the x_2 x_3 plane by 60 degrees.

   b) Find a 3×3 matrix A mapping $\mathbb{R}^3 \to \mathbb{R}^3$ that rotates the x_1 x_3 plane by 60 degrees and leaves the x_2 axis fixed.

7. Find an example of a 2×2 matrix with the property that $A^2 = 0$ but $A \neq 0$.

8. Proof or counterexample. In these L is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, so its representation will be as a 2×2 matrix.

   a) If L is invertible, then L^{-1} is also invertible.

   b) If $LV = 5V$ for all vectors V, then $L^{-1}W = (1/5)W$ for all vectors W.

   c) If L is a rotation of the plane by 45 degrees *counterclockwise*, then L^{-1} is a rotation by 45 degrees *clockwise*.

   d) If L is a rotation of the plane by 45 degrees counterclockwise, then L^{-1} is a rotation by 315 degrees counterclockwise.

   e) The zero map ($0\mathbf{V} := 0$ for all vectors $\mathbf{V}$) is invertible.

   f) The identity map ($I\mathbf{V} := \mathbf{V}$ for all vectors $\mathbf{V}$) is invertible.

   g) If L is invertible, then $L^{-1}0 = 0$.

   h) If $L\mathbf{V} = 0$ for some non-zero vector $\mathbf{V}$, then L is not invertible.

   i) The identity map (say from the plane to the plane) is the only linear map that is its own inverse: $L = L^{-1}$.

9. Let L, M, and P be linear maps from the (two dimensional) plane to the plane:

   L is rotation by 90 degrees counterclockwise.

   M is reflection across the vertical axis

   $Nv := -v$ for any vector $v \in \mathbb{R}^2$ (reflection across the origin)

a) Find matrices representing each of the linear maps L, M, and N.

b) Draw pictures describing the actions of the maps L, M, and N and the compositions: LM, ML, LN, NL, MN, and NM.

c) Which pairs of these maps commute?

d) Which of the following identities are correct—and why?

1)	$L^2 = N$	2)	$N^2 = I$	3)	$L^4 = I$	4)	$L^5 = L$
5)	$M^2 = I$	6)	$M^3 = M$	7)	$MNM = N$	8)	$NMN = L$

10. Think of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as mapping one plane to another.

a) If two lines in the first plane are parallel, show that after being mapped by A they are also parallel – although they might coincide.

b) Let Q be the unit square: $0 < x < 1, 0 < y < 1$ and let Q' be its image under this map A. Show that the area$(Q') = |ad - bc|$. [More generally, the area of any region is magnified by $|ad - bc|$, which is called the *determinant* of A.]

11. Let A be a matrix, not necessarily square. Say $\mathbf{V}$ and $\mathbf{W}$ are particular solutions of the equations $A\mathbf{V} = \mathbf{Y}_1$ and $A\mathbf{W} = \mathbf{Y}_2$, respectively, while $\mathbf{Z} \neq 0$ is a solution of the homogeneous equation $A\mathbf{Z} = 0$. Answer the following in terms of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{Z}$.

a) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1$.

b) Find some solution of $A\mathbf{X} = -5\mathbf{Y}_2$.

c) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

d) Find another solution (other than $\mathbf{Z}$ and 0) of the homogeneous equation $A\mathbf{X} = 0$.

e) Find *two* solutions of $A\mathbf{X} = \mathbf{Y}_1$.

f) Find another solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

g) If A is a square matrix, then $\det A = ?$

h) If A is a square matrix, for any given vector $\mathbf{W}$ can one always find at least one solution of $A\mathbf{X} = \mathbf{W}$? Why?

[Last revised: January 13, 2012]