Math 260, Spring 2012 Jerry L. Kazdan

Homework Set 0 [Due: Never]

Comples Power Series

In our treatment of both differential equations and Fourier series, it will be essential to use comples numbers and complex power series. They enormously simplify the story. This is treated in the Lecture Notes
`http://hans.math.upenn.edu/ kazdan/260S12/notes/math21/math21-2011.pdf` ,
Chapter 0.7, Chapter 1.6.

The guiding force is Euler's use of power series to *define* e^z, where $z = x + iy$ is a complex number by using an analogy with the real power series. Thus, he wrote

$$e^z := 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^k}{k!} + \cdots = \sum_{k=1}^{\infty} \frac{z^k}{k!}.$$

He might have been led to this by observing the close relatonship between the familiar real power series

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots + \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots (-1)^k \frac{x^{2k}}{(2k)!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!} + \cdots \end{aligned}$$

Thus, e^x has all the powers, $\cos x$ only the even powers, and $\sin x$ only the odd powers. The denominators match-up – but then the alternating $\pm$ signs seem mysterious. The mystery symplifies dramatically because Euler had the courage to utilize the sign pattern

$$i^0 = 1, \quad i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \ldots.$$

Then comparing power series we see the remarkable fit in Euler's extrordinary formula

$$e^{it} = \cos t + i \sin t. \tag{1}$$

From this all the properties of the trigonometric functions are consequences of the simpler

$$e^{z+w} = e^z e^w. \tag{2}$$

Example We derive the usual formulas for $\cos(x+y)$ and $\sin(x+y)$. By (2) and then (1)

$$e^{i(x+y)} = e^{ix}e^{iy} = (\cos x + i \sin x)(\cos y + i \sin y) = [\cos x \cos y - \sin x \sin y] + i[\cos x \sin x + \sin x \cos y].$$

But directly using (1),

$$e^{i(x+y)} = \cos(x+y) + i \sin(x+y).$$

Comparing the real and imaginary parts of these last two equations we deduce that

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \quad \text{and} \quad \sin(x+y) = \cos x \sin x + \sin x \cos y.$$

Remarkable.

Notice that we only used the power series for cosine and sine, no triangles. This will be fundamental when we consider radial waves on the surface of a pond. They involve Bessel functions and have no pictorial intrepretation in terms of something elementary such as triangles.

Exercises

The following problems will likely not be obvious to those who have never seen the ideas before. As needed, we will cover the topics in Lecture and Recitation. I hope you find some of these fun — perhaps after struggling. Working with others can be effective.

1. a) Describe (sketch) the real numbers x that satisfy $|x-2| < 3$.
   b) Describe (sketch) the complex numbers z that satisfy $|z-2| < 3$.

2. Write the complex number $2i$ in the *polar form* $2i = re^{i\theta}$, where $r > 0$ and θ is a real angle (in radians). Repeat this for the complex numbers $-2i$ and -1.

3. If $z = x + iy$, where x and y are real, show that $|e^z| = e^x$. In particular, if t is real, $|e^{it}| = 1$.

4. Find the two (complex) roots of $z^2 = i$ (so these are formulas for $\pm\sqrt{i}$, which confused Leibniz). Proceedure: seek z in the polar coordinate form $z = re^{i\theta}$ and note that $i = e^{i\pi/2}$. Thus $z^2 = i$ means $r^2 e^{2i\theta} = e^{i\pi/2}$. Solve this for r and θ. To get *both* roots use $e^{i\theta} = e^{i(\theta+2\pi)}$.

5. [Complex Geometric Series]
   a) If c and z are complex numbers and $n > 0$ an integer, find the usual formula for the sum of the *geometric series*
   $$S_n = c + cz + cz^2 + \cdots + cz^n.$$
   b) Find a formula for $e^{it} + e^{2it} \cdots + e^{int}$.
   c) Find a formula for $\cos t + \cos 2t + \cdots + \cos nt$. [Hint Use (1) and the previous part.]

6. For acomplex-values function $f(t) = u(t) + iv(t)$ of a *real* variable t, define the derivative, $f'(t)$ by the expected rule

$$f'(t) := \lim_{h \to 0} \frac{f(t+h) - f(t)}{h},$$

assuming the limit exits.

   a) If both $u(t)$ and $v(t)$ are differentiable, show that so is f, and then $f'(t) = u'(t) + iv'(t)$ — just as you would guess.

   b) Use this and Euler's foumula (1) to show that for real t, the function $f(t) = e^{it}$ is differentiable with $f'(t) = ie^{it} = if(t)$.

   c) If $c = a + ib$ is a complex constant (a, b real) and $f(t) := e^{ct}$, show that $f'(t) = cf(t)$. Not a surprise.

7. a) Find solutions $u(t)$ of $u'' - u = 0$ in the form $u(t) = e^{rt}$ where r might be a complex number. [Here t is a real variable,]

   b) Find solutions $u(t)$ of $u'' + u = 0$ in the form $u(t) = e^{rt}$ where r might be a complex number. How could you use this to find some real solutions?

   c) Seek (and find) solutions $u(t)$ of $u'' + 2u' + 5u = 0$ in the form $u(t) = e^{rt}$ where r might be a complex number. How could you use this to find some real solutions?

Equations as Maps

The next set of problems have an entirely different character. They are about viewing systems of equations as maps. Think of this an an introduction to *computer graphics.* We'll use these ideas throughout Math 260.

The standard technique goes back to Descartes' introduction of coordinates in geometry. Say one has two copies of the plane, the first with coordinates (x_1, x_2), the second with coordinates (y_1, y_2). Then the high school equations

$$x_1 - 0x_2 = y_1 \tag{3}$$

$$x_1 + x_2 = y_2 \tag{4}$$

can be thought of as a mapping from the (x_1, x_2) plane to the (y_1, y_2) plane. For instance, if $x_1 = 1$ and $x_2 = 0$, then $y_1 = 1$ and $y_2 = 1$. Thus the point $(1, 0)$ is mapped to the point $(1, 1)$.

1. a) What are the images of $(1, 0)$, $(1, \pi/4)$, and $(0, \pi/4)$?

   b) What is the image of the rectangle with vertices at $(0, 0)$, $(1, 0)$, $(1, \pi/4)$, and $(0, \pi/4)$? [Draw a sketch.]

   c) What is the image of the rectangle with vertices at $(1, 0)$, $(2, 0)$, and $(2, \pi/4)$, and $(1, \pi/4)$?

2. Next consider the *nonlinear* map from the (x_1, x_2) plane to the (y_1, y_2) plane

$$x_1 \cos x_2 = y_1 \tag{5}$$
$$x_1 \sin x_2 = y_2 \tag{6}$$

   a) What are the images of $(1, 0)$, $(1, \pi/4)$, and $(0, \pi/4)$?
   b) What is the image of the rectangle with vertices at $(0, 0)$, $(1, 0)$, $(1, \pi/4)$, and $(0, \pi/4)$? [Draw a sketch.]
   c) What is the image of the rectangle with vertices at $(1, 0)$, $(2, 0)$, and $(2, \pi/4)$, and $(1, \pi/4)$?

3. Consider the linear map from the (x_1, x_2) plane to the (y_1, y_2) plane defined by the equations

$$ax_1 + bx_2 = y_1$$
$$cx_1 + dx_2 = y_2,$$

   where the (real) coefficients satisfy $ad - bc \neq 0$. Show that the image of a straight line ℓ in the (x_1, x_2) plane is a straight line in the (y_1, y_2) plane, and that this image contains the origin if and only if ℓ contains the origin.

[Last revised: January 10, 2012]