Problem Set 12 Solutions

1. Let $\gamma(t)$ be any smooth closed curve in $\mathbb{R}^4$. Why, with only a quick mental computation, is $\oint_\gamma 2x\,dx + 6(x-y)\,dy = \oint_\gamma 6x\,dy$?

Note that

$$\oint_\gamma 2x\,dx + 6(x-y)\,dy = \oint_\gamma 2x\,dx + 0\,dy + \oint_\gamma 0\,dx - 6y\,dy + \oint_\gamma 0\,dx + 6x\,dy .$$

The first two are conservative vector fields - gradients are possibly emerging from the potentials x^2 and $-3y^2$, respectively. As γ is closed, the first two path integrals are zero. We can conclude here, but we confirm that $(0, 6x, 0, 0)$ is not conservative:

If it were, $\exists$ a scalar-valued function φ s.t $\nabla\varphi = (0, 6x, 0, 0)$. We assume φ is smooth, and so $\frac{d^2\varphi}{dx\,dy} = 6 \neq 0 = \frac{d}{dy}(0) = \frac{d^2\varphi}{dy\,dx}$ means that this was a pipe dream. So, our original path integral $= \oint_\gamma 6x\,dy$.

2. Find some closed curve $\gamma(t)$ s.t $\oint_\gamma 6x\,dy > 0$.

Let $\gamma(t) = (\cos t, \sin t)$ for $0 \le t \le 2\pi$.

$\gamma'(t) = (-\sin t, \cos t)$ so

$$\oint_\gamma 6x\,dy = \int_0^{2\pi} 6(\cos t)(\cos t)\,dt = \int_0^{2\pi} 6\cos^2 t > 0$$ as $\cos t \neq$ the zero function from 0 to 2π.

3. Let $C :=$ portion of $x^2+y^2=1$ with $x \geq 0$ oriented so that it begins at $(0,1)$ and ends at $(0,-1)$. Evaluate

$$\int_C e^x \sin y \, dx + e^x \cos y \, dy .$$

Note that $\int e^x \sin y \, dx = e^x \sin y + h(y)$,

$\int e^x \cos y \, dy = e^x \sin y + g(x)$ so

letting $g(x) = h(y) = a$, we see that our vector field is conservative with potential $e^x \sin y + a$, $a \in \mathbb{R}$.

By the "fundamental thm. of line integrals",

$$\int_C e^x \sin y \, dx + e^x \cos y \, dy = \int_C \nabla (e^x \sin y + a) \cdot dr = e^x \sin y + a \Big]_{(0,-1)}^{(0,1)}$$

$$= (e^0 \sin 1 + a) - (e^0 \sin(-1) + a) \quad [\sin(x) \text{ is odd, so}]$$

$$= e^0 \sin 1 + e^0 \sin 1 + a - a$$

$$= \boxed{2 \sin 1}$$

4. $$\int_a^b F \cdot X'(t) \, dt = \int_a^b F(X(t)) \cdot X'(t) \, dt \quad [\text{unsuppressing notation}]$$

$$= \int_a^b m X''(t) \cdot X'(t) \, dt \quad [\text{Newton's } 2^{nd}]$$

$$= \Big[m X'(t) \cdot X'(t) \Big]_a^b - m \int_a^b X'(t) \cdot X''(t) \, dt \quad \left[\begin{matrix}\text{Integration by} \\ \text{parts}\end{matrix}\right]$$

so $2m \int_a^b X''(t) \cdot X'(t) \, dt = \Big[m X'(t) \cdot X'(t) \Big]_a^b$

$$\Rightarrow \int_a^b F(X(t)) \cdot X'(t) \, dt = \frac{m}{2} \|X'(b)\|^2 - \frac{m}{2} \|X'(a)\|^2 .$$

Work - KE Equivalence - thanks to integration by parts.

5. Assume there exists a scalar-valued $\varphi(r)$ s.t $\psi(\|X\|)\, X = F(X) = \nabla \varphi(\|X\|)$.

Compute $\nabla \varphi(\|X\|)$: [Remember, we are in $\mathbb{R}^3$]

for example, $\frac{d\varphi}{dx}(\|X\|) = \frac{d\varphi}{d\|X\|}\frac{d\|X\|}{dx} = \varphi'(\|X\|)\frac{d}{dx}\left[\sqrt{x^2+y^2+z^2}\right]$

$$= \varphi'(\|X\|)\frac{x}{\sqrt{x^2+y^2+z^2}} \quad \text{so}$$

$$\nabla \varphi(\|X\|) = \frac{\varphi'(\|X\|)}{\|X\|}(x,y,z) = \frac{\varphi'(\|X\|)}{\|X\|}X.$$

so $\psi(\|X\|)\cdot\|X\| = \varphi'(\|X\|)$ and letting $\|X\| = r$, we suggest that

$$\varphi(r) = \int_0^r \psi(y)\, y\, dy.$$

Check: $\nabla\varphi(\|X\|)$:

again, $\frac{d\varphi}{dx}(\|X\|) = \frac{d\varphi}{d\|X\|}\frac{d\|X\|}{dx}$.

By FTC: $\frac{d\varphi}{dr}(r) = \frac{d}{dr}\int_0^r \psi(y)\, y\, dy = \psi(r)\, r$ so

$$\frac{d\varphi}{dx}(\|X\|) = \psi(\|X\|)\,\|X\|\,\frac{x}{\|X\|} = \psi(\|X\|)\, x \quad \text{so}$$

$$\nabla\varphi(\|X\|) = \psi(\|X\|)\,(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= \psi(\|X\|)\, X.$$

As ψ is C^1 from 0 to ∞, φ is at least C^2 (away from 0) $(r>0)$.

6. Our surface is defined in $\mathbb{R}^3$ by

$$x = u^2 - v^2, \quad y = u + v, \quad z = u^2 + 4v$$

a) The surface is regular when $T_u \times T_v \neq 0$.

$$T_u = \frac{\partial x}{\partial u}(u,v)\,\hat{\imath} + \frac{\partial y}{\partial u}(u,v)\,\hat{\jmath} + \frac{\partial z}{\partial u}(u,v)\,\hat{k}$$

$$= (2u, 1, 2u)$$

$$T_v = \frac{\partial x}{\partial v}(u,v)\,\hat{\imath} + \frac{\partial y}{\partial v}(u,v)\,\hat{\jmath} + \frac{\partial z}{\partial v}(u,v)\,\hat{k}$$

$$= (-2v, 1, 4)$$

$$T_u \times T_v = (4-2u)\hat{\imath} + (-8u - 4uv)\hat{\jmath} + (2u+2v)\hat{k}$$

If $T_u \times T_v = 0$,

$4 = 2u \Rightarrow u = 2$, $\quad -8(2) - 4(2)(-2) = -16 + 16 = 0$ ✓

$2u + 2v = 0 \Rightarrow v = -2$,

$\therefore$ regular when $(u,v) \neq (2,-2)$ ~~or [illegible]~~
or when $(x,y,z) \neq (0,0,-4)$

b) Find the equation of the tangent plane at $(-1/4, 1/2, 2)$ $[u=0, v=1/2]$.

As the surface is regular here, T_u spans a plane along with T_v, and we can ~~be~~ define the tangent plane as span$\{T_u, T_v\}$.

If $n = T_u \times T_v$, n is a normal vector to the tangent plane,

so we can give an equation for the plane:

$$n \cdot (x - (-1/4), y - 1/2, z - 2) = 0.$$

$$n = (4 - 2u, -8u - 4uv, 2u + 2v)\big|_{(0, 1/2)} = (4, 0, 1)$$

so the tangent plane is

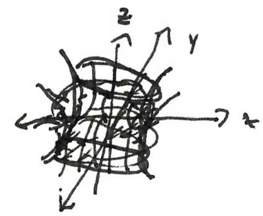
~~4x~~ $4x + 1 + 0(y - 1/2) + z - 2 = 0$

$$\boxed{4x + z = 1}$$

7.

a) $\Phi(u,v) = (2\sqrt{1+u^2}\cos v,\ 2\sqrt{1+u^2}\sin v,\ u)$

is a hyperboloid

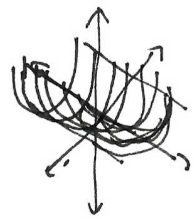


b) $\Phi(u,v) = (3\cos u \sin v,\ 2\sin u \sin v,\ \cos v)$

is an ellipsoid

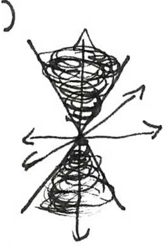
c) $\Phi(u,v) = (u,\ v,\ u^2)$

is a parabolic cylinder



d) $\Phi(u,v) = (u\cos v,\ u\sin v,\ u)$

is a cone



8. For a sphere in $\mathbb{R}^3$ centered at $(0,0,0)$ with radius 2, find the equation of the tangent plane at $\underline{(1,\ 1,\ \sqrt{2})}$ by considering the sphere as

a) $\Phi(\theta,\phi) := (2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi)$

$\Phi_\theta = \cancel{(2\cos\theta} \ (-2\sin\theta\sin\phi,\ 2\cos\theta\sin\phi,\ 0)$

$\Phi_\phi = (2\cos\theta\cos\phi,\ 2\sin\theta\cos\phi,\ -2\sin\phi)$

$\cancel{\Phi_\theta \times}$

$1 = 2\cos\theta\sin\phi,$

$1 = 2\sin\theta\sin\phi, \quad \Rightarrow \quad \phi = \pi/4,\ \theta = \pi/4.$

$\sqrt{2} = 2\cos\phi,$

so $\Phi_\theta = (-2\cdot 1/\sqrt{2}\cdot 1/\sqrt{2},\ 2\cdot 1/\sqrt{2}\cdot 1/\sqrt{2},\ 0)$

$= (-1,\ 1,\ 0)$

$\Phi_\phi = (1,\ 1,\ -\sqrt{2})$

so $\Phi_\theta \times \Phi_\phi = (-\sqrt{2},\ -\sqrt{2},\ -2) = n,$

$n \cdot (x-1,\ y-1,\ z-\sqrt{2}) = 0$

$\Rightarrow -\sqrt{2}(x-1) - \sqrt{2}(y-1) - 2(z-\sqrt{2}) = 0$

$\sqrt{2}(x-1) + \sqrt{2}(y-1) + 2(z-\sqrt{2}) = 0$

$$\boxed{\sqrt{2}x + \sqrt{2}y + 2z = 4\sqrt{2}}$$

8b) as a level surface of $x^2+y^2+z^2 = f(x,y,z)$.

$$\nabla f(x,y,z)\big|_{(1,1,\sqrt{2})} = (2x, 2y, 2z)\big|_{(1,1,\sqrt{2})} = (2,2,2\sqrt{2})$$

tangent plane: $\nabla f \cdot (x-x_0, y-y_0, z-z_0) = 0$

so $(2,2,2\sqrt{2})\cdot(x-1, y-1, z-\sqrt{2}) = 0$

$$\boxed{2x+2y+2\sqrt{2}z = 8}$$ (same as a))

c) as the graph of $g(x,y) := \sqrt{4-x^2-y^2}$.

$x=u=1, y=v=1, \sqrt{4-1-1} = \sqrt{2} = z$)

Our surface is $\Phi(u,v) = (u, v, \sqrt{4-u^2-v^2})$

$$\Phi_u = \left(1, 0, \frac{-u}{\sqrt{4-u^2-v^2}}\right)_{(1,1)} = (1, 0, -1/\sqrt{2})$$

$$\Phi_v = \left(0, 1, \frac{-v}{\sqrt{4-u^2-v^2}}\right) = (0, 1, -1/\sqrt{2})$$

$$\Phi_u \times \Phi_v = (1/\sqrt{2}, 1/\sqrt{2}, 1) = n$$

$$n\cdot(x-1, y-1, z-\sqrt{2}) = 0$$

$$\Rightarrow 1/\sqrt{2}(x-1) + 1/\sqrt{2}(y-1) + z - \sqrt{2} = 0$$

$$\boxed{1/\sqrt{2}\,x + 1/\sqrt{2}\,y + z = 2\sqrt{2}}$$ (same as a and b, of course.)

9) $x(\theta,\phi) = (3+\cos\phi)\cos\theta$
$y(\theta,\phi) = (3+\cos\phi)\sin\theta$
$z(\theta,\phi) = \sin\phi$, $0 \le \theta, \phi \le 2\pi$.

Show that the image surface is regular at all points.

$$T_\theta = (-\sin\theta(3+\cos\phi), \cos\theta(\cos\phi+3), 0)$$

$$T_\phi = (-\cos\theta\sin\phi, -\sin\theta\sin\phi, \cos\phi)$$

$$T_\theta \times T_\phi = \big(\cos\theta\cos\phi(\cos\phi+3), \cos\phi\sin\theta(3+\cos\phi),$$
$$[\sin^2\theta + \cos^2\theta]\sin\phi(\cos\phi+3)\big)$$

$$= (\cos\theta\cos\phi(\cos\phi+3), \cos\phi\sin\theta(3+\cos\phi), \sin\phi(\cos\phi+3)).$$

If $T_\theta \times T_\phi = 0$, $\sin\phi(\cos\phi+3) = 0$ but $-1 \le \cos\phi \le 1$ so $\sin\phi = 0 \Rightarrow \cos\phi \ne 0$ so $\cos\theta = \sin\theta = 0$ $\times$ Contradiction

$\boxed{\therefore \text{It's regular at all points.}}$

10) Compute the surface area of the torus from #9, using the parametrization from #9.

$$\overline{SA}(\text{torus}) = \iint_D \| T_\theta \times T_\phi \| \, d\theta \, d\phi \qquad D = (\theta, \phi) \in [0, 2\pi] \times [0, 2\pi]$$

$$\| T_\theta \times T_\phi \| = \sqrt{\cos^2\theta(\cos^2\phi(\cos\phi+3)^2) + \sin^2\theta(\cos^2\phi(\cos\phi+3)^2) + \sin^2\phi(\cos\phi+3)^2}$$

$$= \sqrt{\cos^2\phi(\cos\phi+3)^2 + \sin^2\phi(\cos\phi+3)^2}$$

$$= \sqrt{(\cos\phi + 3)^2}$$

$$= 3 + \cos\phi$$

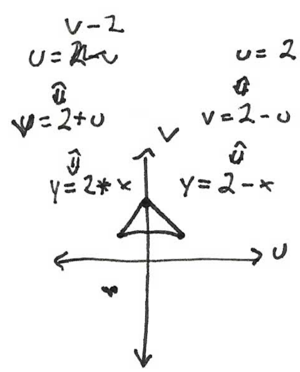
$$SA(\text{torus}) = \int_0^{2\pi}\int_0^{2\pi} 3 + \cos\phi \, d\theta \, d\phi$$

$$= 2\pi \int_0^{2\pi} 3 + \cos\phi \, d\phi = 2\pi \left(3\phi \Big]_0^{2\pi} + \sin\phi \Big]_0^{2\pi} \right)$$

$$= 2\pi \cdot 3 \cdot 2\pi$$

$$= \boxed{12\pi^2}$$

11) Consider the graph of $z := y^3 \cos^2 x$ over the triangle w/vertices at $(-1,1)$, $(0,2)$, $(1,1)$. Express the surface area as an integral, don't evaluate it.



Phrase our surface as $(u, v, \cos^2 u \cdot v^3)$ and

$$\| T_u \times T_v \| = \sqrt{1 + \left(\frac{dz}{du}\right)^2 + \left(\frac{dz}{dv}\right)^2}$$

$$\text{so } SA = \iint_{\text{triangle}} \sqrt{1 + 4v^6 \sin^2 u \cos^2 u + 9v^4 \cos^4 u} \, du \, dv$$

$$\boxed{= \int_1^2 \int_{v-2}^{-v+2} \sqrt{1 + 4v^6 \sin^2 u \cos^2 u + 9v^4 \cos^4 u} \, du \, dv}$$

12) Evaluate the integral $\iint_S (x+z)\, dS$, where S is the part of the cylinder $y^2+z^2=4$ with $0 \le x \le 5$.

We parametrize:

$$\begin{aligned} x &\to x \\ y &\to r\cos\theta \quad , \; r=2. \\ z &\to r\sin\theta \end{aligned}$$

$$T_x = (1, 0, 0)$$
$$T_\theta = (0, -2\sin\theta, 2\cos\theta)$$
$$T_x \times T_\theta = \cancel{(-2\sin\theta, -2\cos\theta, 0)}$$
$$= (0, -2\cos\theta, -2\sin\phi)$$
$$|T_x \times T_\theta| = \sqrt{4\cos^2\theta + 4\sin^2\theta} = 2.$$

So as

$$\iint_S (x+z)\, dS = \iint_{S^*} (x + 2\sin\theta)\cdot 2 \cdot dx\, d\theta$$
$$= \int_0^5 \int_0^{2\pi} 2x + 4\sin\theta \; d\theta\, dx$$
$$= 2\pi \int_0^5 2x\, dx = 2\pi \cdot 10 = \underline{20\pi}$$

(§7.5: $\iint_S f(x,y,z)\, dS = \iint_D f(\Phi(u,v))\, \|T_u \times T_v\|\, du\, dv$

where $\Phi : \underset{\mathbb{R}^2}{\overset{\cap}{D}} \to S \subset \mathbb{R}^3$, D is elementary, f is a real-valued continuous function defined on S)

13) Let S be a two-dimensional surface in $\mathbb{R}^n$, $n \geq 3$, given by the parametrization $(u,v) \to \Phi(u,v)$ with

$x_1 = x_1(u,v)$, $x_2 = x_2(u,v)$, $\ldots$, $x_n = x_n(u,v)$.

Say we have a curve $\gamma(t) = (u(t), v(t))$. Then $X(t) := \Phi(u(t), v(t))$ is a curve in the surface in $\mathbb{R}^n$.

If $ds = \|\gamma'(t)\|\, dt$, show that

$$\left(\frac{ds}{dt}\right)^2 = E(u,v)\left(\frac{du}{dt}\right)^2 + 2F(u,v)\frac{du}{dt}\frac{dv}{dt} + G(u,v)\left(\frac{dv}{dt}\right)^2$$

where $E(u,v) = \left\|\frac{d\Phi}{du}\right\|^2$, $F(u,v) = \frac{d\Phi}{du}\cdot\frac{d\Phi}{dv}$, $G(u,v) = \left\|\frac{d\Phi}{dv}\right\|^2$.

$$\left(\frac{ds}{dt}\right)^2 = \|\gamma'(t)\|^2 = \langle \gamma'(t), \gamma'(t)\rangle.$$

We must express γ in terms of its image in the surface: $\Phi(\gamma(t)) = X(t)$.

$$X'(t) = \frac{d\Phi}{du}\frac{du}{dt} + \frac{d\Phi}{dv}\frac{dv}{dt} \quad \text{(chain rule) and so}$$

$$\langle X'(t), X'(t)\rangle = \left\langle \frac{d\Phi}{du}\frac{du}{dt} + \frac{d\Phi}{dv}\frac{dv}{dt}, \frac{d\Phi}{du}\frac{du}{dt} + \frac{d\Phi}{dv}\frac{dv}{dt}\right\rangle$$

$$= \left\langle \frac{d\Phi}{du}, \frac{d\Phi}{du}\right\rangle \frac{du}{dt}\frac{du}{dt} + 2\left\langle \frac{d\Phi}{du}, \frac{d\Phi}{dv}\right\rangle \frac{du}{dt}\frac{dv}{dt} + \left\langle \frac{d\Phi}{dv}, \frac{d\Phi}{dv}\right\rangle \frac{dv}{dt}\frac{dv}{dt}$$

$$= E(u,v)\left(\frac{du}{dt}\right)^2 + 2F(u,v)\frac{du}{dt}\frac{dv}{dt} + G(u,v)\left(\frac{dv}{dt}\right)^2.$$