Math 260

Solutions, Problem Set 9

1. a) Find local extrema of $f(x,y) := 3x+4y$, $x^2+y^2 < 1$.

- First, we look for critical points: $\nabla f(x,y) = 0$ there.

$\nabla f = (3,4) \neq (0,0)\ \forall x,y$ in our region, so there are no local extrema.

b) Now, find local extrema of $f(x,y)$ on $x^2+y^2=1$.

We use Lagrange multipliers: $g(x,y) = x^2+y^2$.

Seek $\hat{(x,y)}, \lambda$ s.t. $\nabla f = \lambda \nabla g$: $(\lambda \neq 0)$

$$(3,4) = \lambda(2x, 2y)$$

$$2\lambda x = 3, \quad 2y\lambda = 4 \implies x = \frac{3}{2\lambda}, \quad y = \frac{4}{2\lambda}$$

$$1 = x^2+y^2 = \left(\frac{3}{2\lambda}\right)^2 + \left(\frac{4}{2\lambda}\right)^2 = \frac{25}{4\lambda^2} \implies \lambda = \pm 5/2.$$

$\lambda = 5/2 \implies x = 3/5,\ y = 4/5$ and $\lambda = -5/2 \implies x = -3/5,\ y = -4/5$.

(Some people tried to use bordered Hessians to handle the analysis from here - suffice it to say, they're a bit esoteric/involved and we've never really talked about them.) (They're okay to use - see Thm 10, pg. 198)

at $(3/5, 4/5)$, $f(x,y) = 5$, and at $(-3/5, -4/5) \implies f(x,y) = -5$.

$\therefore$ f has a local max at $(3/5, 4/5)$ (5)
" " " " min at $(-3/5, -4/5)$ (-5)

c) Find global maxima and minima of f on $x^2+y^2 \leq 1$.

By the Extreme Value Thm (see Marsden + Tromba §3.3, Thm 7. hypothesis: region is closed and bounded, which for those interested, implies compact in $\mathbb{R}^n$ with the standard metric), global maxima and minima exist in our region.

There are no local maxima/minima when $x^2+y^2 \leq 1$, so our only candidates are our local extrema on $x^2+y^2=1$.

$\therefore$ Global max of 5 at $(3/5, 4/5)$,
Global min of -5 at $(-3/5, -4/5)$.

2. Let x, y, z be positive reals such that $x+y+z=a$, a is a fixed positive real. How large can xyz be?

Let $f(x,y,z) = xyz$, $g(x,y,z) = x+y+z$. We maximize f with the constraint $g = a > 0$. We seek x, y, z, λ s.t. $\lambda \neq 0$ and

$$\nabla f = \lambda \nabla g$$

$$(yz, xz, xy) = \lambda(1,1,1)$$

$$yz = \lambda = xz = xy. \qquad (xz = \lambda = xy, \quad x > 0 \text{ so})$$

$$z = \frac{\lambda}{x} = y \text{ and similarly}$$

$$x = y = z, \quad x+y+z = a \quad \text{so}$$

$$x = y = z = a/3, \quad \lambda = a^2/9 \quad \text{and}$$

$$f(a/3, a/3, a/3) = a^3/27.$$

This is a maximum: I'll eat my words and form the bordered Hessian: let $h = f - \lambda(g - a)$

$$|\bar{H}_3| = \begin{vmatrix} 0 & -\frac{\partial g}{\partial x} & -\frac{\partial g}{\partial y} & -\frac{\partial g}{\partial z} \\ -\frac{\partial g}{\partial x} & \frac{\partial^2 h}{\partial x^2} & \partial^2 h/\partial x \partial y & \partial^2 h/\partial x \partial z \\ -\frac{\partial g}{\partial y} & \partial^2 h/\partial x \partial y & \frac{\partial^2 h}{\partial y^2} & \partial^2 h/\partial y \partial z \\ -\frac{\partial g}{\partial z} & \partial^2 h/\partial x \partial z & \partial^2 h/\partial x \partial y & \frac{\partial^2 h}{\partial z^2} \end{vmatrix}_{\substack{\text{at } (a/3, a/3, a/3) \\ \lambda = a^2/9}}$$

$$= \begin{vmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & z & y \\ -1 & z & 0 & x \\ -1 & y & x & 0 \end{vmatrix} = \begin{array}{l} 0 \\ -(-1)(x^2 - xz - yz) \\ +(-1)(xy - 0 + yz - y^2) \\ -(1)(-zx - 0 + z^2 - zy) \\ = -a^2/9 < 0 \end{array}$$

$$|\bar{H}_2| = \begin{vmatrix} 0 & -\partial g/\partial x & -\partial g/\partial y \\ -\partial g/\partial x & \partial^2 h/\partial x^2 & \partial^2 h/\partial x \partial y \\ -\partial g/\partial y & \partial^2 h/\partial x \partial y & \partial^2 h/\partial y^2 \end{vmatrix}_{\substack{\text{at } (a/3, a/3, a/3) \\ \lambda = a^2/9}}$$

$$\begin{vmatrix} 0 & -1 & -1 \\ -1 & 0 & z \\ -1 & z & 0 \end{vmatrix} = z + z = 2a/3 > 0$$

so we have a local max (see pg. 199)

(This is why we don't make you really do this!)

$$\boxed{\therefore\ xyz \le a^3/27, \text{ with equality achieved.}}$$

b) If $x, y, z \ge 0$, show that $(xyz)^{1/3} \le \frac{x+y+z}{3}$.

If x, y, or $z = 0$, $(xyz)^{1/3} = 0 \le \frac{x+y+z}{3}$.

If all are positive, $(xyz)^{1/3} \le \left(a^3/27\right)^{1/3} = a/3 = \frac{x+y+z}{3}$. ∎

Why? $\nabla f = (yz, xz, xy) \neq (0,0,0)$ so the maximum we found was the global one in the domain $x \ge 0$, $y \ge 0$, $z \ge 0$, $x+y+z \le a$ for any fixed positive a (Extreme Value Thm.)

3. Let $f(x,y) := [x^2 + (y+2)^2][x^2 + (y-2)^2]$.

a) Classify / find all critical points of f in $\mathbb{R}^2$.

$\nabla f = 0$: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$\frac{\partial f}{\partial x} = 2x(x^2 + (y-2)^2) + 2x(x^2+(y+2)^2) = 4x(x^2+y^2+4) = 0$ only when $x = 0$, y = anything, as (x^2+y^2+4)

$\frac{\partial f}{\partial y} = 4y(x^2+y^2-4) = 0$ if $y = 0$, x = anything or $x = 0$, $y = \pm 2$ or $x^2+y^2 = 4$ ... but combining w/above,

Only overlap is $(0,0)$, $(0,2)$, $(0,-2)$

~~b)~~ Now classify with the Hessian:

$$Hf \begin{vmatrix} 4(3x^2+y^2+4) & 8xy \\ 8xy & 4(x^2+3y^2-4) \end{vmatrix}$$

At $(0,0)$, $Hf = \begin{vmatrix} 16 & 0 \\ 0 & -16 \end{vmatrix} \Rightarrow$ saddle point. (indef.)

At $(0,2)$, $Hf = \begin{vmatrix} 32 & 0 \\ 0 & 32 \end{vmatrix} \Rightarrow$ local min. (pos. def)

At $(0,-2)$ $Hf = \begin{vmatrix} 32 & 0 \\ 0 & 32 \end{vmatrix} \Rightarrow$ local min. (pos. def)

$\therefore$ critical points	behavior	value
$(0,0)$	saddle	16
$(0,2)$	local min.	0
$(0,-2)$	local min.	0

b) Find the max/min of f on $x^2+y^2 = 1$.

Let $g(x,y) = x^2+y^2$.

Seek (x,y), $\lambda \neq 0$ s.t $\nabla f = \lambda \nabla g$

$4x(x^2+y^2+4) = \lambda(2x)$
$4y(x^2+y^2-4) = \lambda(2y)$
$\Rightarrow$
$4x(5) = \lambda(2x)$ $\quad x = 0$ or $\lambda = 10$
$4y(-3) = \lambda(2y)$ $\quad y = 0$ or $\lambda = -6$

only possible pairs: $(x=0, y=\pm 1, \lambda = -6)$
$(x = \pm 1, y = 0, \lambda = 10)$

values: $(0,1)$: 9
$(0,-1)$: 9 $\Big\}$ min.
$(1,0)$: 25
$(-1,0)$: 25 $\Big\}$ max.

c) Find global maxima / minima in $x^2+y^2 \leq 1$.

By the Extreme Value Thm: they exist, and by a + b the only critical points in $x^2+y^2 < 1$ ~~are~~ is $(0,0)$: value, 16 so as $16 > 9$, $16 < 25$

$\therefore$ global maxima are $(1,0)$, $(-1,0)$
global minima are $(0,1)$, $(0,-1)$.

d) Same analysis as in b), but look at $x^2+y^2 = 9$:

critical points $(\pm 3, 0)$, $(0, \pm 3)$
values 169 25 so from a,

global maxima are $(\pm 3, 0)$, value: 169, global minima is at $(0, \pm 2)$ value: 0

4. Find the max/min value of $h(x,y,z) = (x-y)^2 + z^2$ on $x^2+y^2+z^2 = 18$.

Lagrange multipliers: $g(x,y,z) = x^2+y^2+z^2$.

$$\nabla f = \lambda \nabla g : (2x-2y,\ 2y-2x,\ 2z) = \lambda(2x, 2y, 2z)$$

either $\lambda = 1$, $x=y=0$, $z \in \mathbb{R}$ or

$x = \pm y$, $\lambda = 2$, $z = 0$.

constraint: $(0, 0, \pm\sqrt{18})$, $(\pm 3, \pm 3, 0)$, $(\pm 3, \mp 3, 0)$.

values: 18 values: 0 values: 36

$$\boxed{\begin{array}{ll} \text{min: } (3,3,0),\ (-3,-3,0) & \text{value: } 0 \\ \text{max: } (3,-3,0),\ (-3,3,0) & \text{value: } 36 \end{array}}$$

5. in $\mathbb{R}^2$: $r^2 = x^2+y^2$, Assume $u(x,y) = w(r)$ for some smooth function w.

a) Show that $\frac{du}{dx} = \frac{dw}{dr}\frac{x}{r}$.

$$\frac{du}{dx} = \frac{dw}{dr}\frac{dr}{dx}, \quad \frac{dr}{dx}: \quad 2r\frac{dr}{dx} = 2x \Rightarrow \frac{dr}{dx} = \frac{x}{r}.$$

$$\boxed{\therefore \frac{du}{dx} = \frac{dw}{dr}\frac{x}{r}.}$$

b) Compute $\frac{d^2u}{dx^2}$:

$$= \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{d}{dx}\left(\frac{dw}{dr}\frac{x}{r}\right) = \frac{d}{dr}\left(\frac{dw}{dr}\frac{x}{r}\right)\frac{dr}{dx}$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right) x \cdot \frac{1}{r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial r}\frac{\partial x}{\partial r} \cdot \frac{1}{r} \cdot \frac{\partial r}{\partial x} + \frac{\partial w}{\partial r} \times \frac{\partial}{\partial r}\left(\frac{1}{r}\right) \cdot \frac{\partial r}{\partial x}.$$

Now $\frac{\partial x}{\partial r} = \frac{1}{\frac{\partial r}{\partial x}}$ because $(x,y) = x^2+y^2 = r^2$ is an $\mathbb{R}^2 \to \mathbb{R}^1$ function and we can apply the Inverse function Thm. (Thm 11, § 3.5, pg. 204, Marsden + Tromba)

Not true in general!

$$\text{so } \frac{d^2u}{dx^2} = \frac{d^2w}{dr^2}\frac{x^2}{r^2} + \frac{dw}{dr}\cdot\frac{1}{r} + \frac{dw}{dr} x \left(-\frac{1}{r^2}\right)\frac{x}{r}$$

$$\boxed{= \frac{d^2w}{dr^2}\left(\frac{x^2}{r^2}\right) + \left(\frac{1}{r} - \frac{x^2}{r^3}\right)\frac{dw}{dr}}$$

c) Show that $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = \frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}$.

By symmetry, $\frac{d^2u}{dy^2} = \frac{d^2w}{dr^2}\left(\frac{y^2}{r^2}\right) + \left(\frac{1}{r} - \frac{y^2}{r^3}\right)\frac{dw}{dr}$ and

$$\frac{d^2u}{dy^2} + \frac{d^2u}{dx^2} = \frac{d^2w}{dr^2}\left(\frac{x^2+y^2}{r^2}\right) + \left(\frac{2}{r} - \frac{(x^2+y^2)}{r^3}\right)\frac{dw}{dr} \qquad \left(\frac{2}{r} - \frac{(x^2+y^2)}{(x^2+y^2)^{3/2}} = \frac{2}{r} - \frac{1}{r} = \frac{1}{r}\right)$$

$$\boxed{= \frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}}$$

d) Find all radial functions $u(x,y)$ s.t. $\Delta u = u_{xx} + u_{yy} = 0$.

From a) - c): $\Delta u = \frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}$.

Let $v(r) = \frac{dw}{dr}$ so if $\Delta u = 0$, $\frac{dv}{dr} = -\frac{v(r)}{r}$

$$\Rightarrow \frac{dv}{-v} = \frac{dr}{r} \Rightarrow \ln(-v) = \ln(r) + C$$

$$\Rightarrow \ln(1/v) = \ln(r) + C$$

$$1/v = Ar$$

$$v(r) = \frac{B}{r} \qquad (A, B, C, D \text{ constants})$$

so $\boxed{w(r) = B\ln(r) + D}$ is our complete classification $\left(\text{or } 2B\ln(x^2+y^2)+D\right)$

e) Generalize to 3 dimensions:

$$u_{xx} + u_{yy} + u_{zz} = \frac{d^2 w}{dr^2}\left(\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}\right) + \frac{dw}{dr}\left(\frac{1}{r} - \frac{x^2}{r^3} + \frac{1}{r} - \frac{y^2}{r^3} + \frac{1}{r} - \frac{z^2}{r^3}\right)$$

$$= \frac{d^2 w}{dr^2} + \frac{dw}{dr}\left(\frac{2}{r}\right)$$

so if $u_{xx} + u_{yy} + u_{zz} = 0$,

$\frac{d^2 w}{dr^2} = -\frac{2}{r}\frac{dw}{dr}$ and if $v(r) = \frac{dw}{dr}$,

$\frac{1}{v}\frac{dv}{dr} = \frac{-2}{r} \Rightarrow \frac{C}{r^2}$ and

$$\boxed{w(r) = -\frac{C}{r} + D \qquad (C, D \text{ constants})}$$

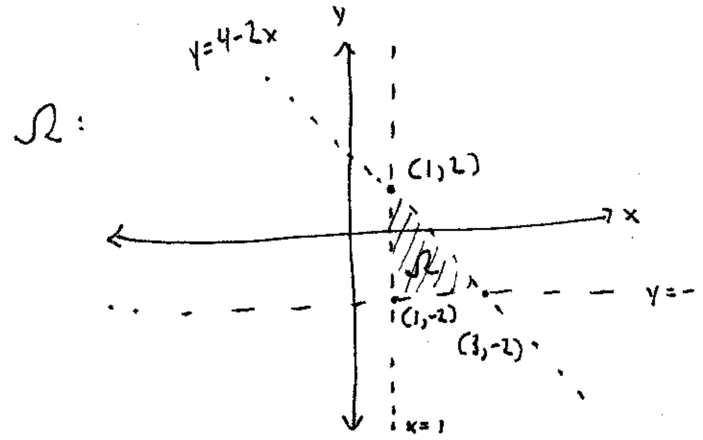
6) I wanted to see a connection between the integral and Riemann sums. Anything along these lines of thought was acceptable.

7) Volume: $\displaystyle\int_{-1}^{5}\int_{-2}^{2} (f(x,y)+3) - (f(x,y)-1)\, dy\, dx$

$$= \int_{-1}^{5}\int_{-2}^{2} 4 \, dy\, dx = \int_{-1}^{5} \left[4y\right]_{-2=y}^{2=y} dx = \int_{-1}^{5} 16\, dx = 16x\Big]_{x=-1}^{x=5} = \boxed{96}$$

8. Let $J = \iint_{\Omega} (x-2y)^2 \, dA$, $\Omega :=$ △ bounded by
$x = 1$
$y = -2$
$y + 2x = 4$.

Ω:



bounds are $x = \frac{-y+4}{2}$ and $x = 1$, $-2 \le y \le 2$

a) Evaluate J by integrating w.r.t. x first:

$$J = \int_{-2}^{2}\int_{1}^{\frac{4-y}{2}} (x-2y)^2 \, dx \, dy = \int_{-2}^{2}\int_{1}^{\frac{4-y}{2}} x^2 - 4xy + 4y^2 \, dx \, dy$$

$$= \int_{-2}^{2} \left[\frac{x^3}{3} - 2x^2y + 4y^2x \right]_{1}^{\frac{4-y}{2}} dy$$

$$= \int_{-2}^{2} \frac{(4-y)^3}{24} - \frac{2(4-y)^2}{4}y + 2\frac{y^2(4-y)}{2} - \left(\frac{1}{3} - 2y + 4y^2\right) dy$$

$$= \int_{-2}^{2} -\frac{61y^3}{24} + \frac{17y^2}{2} - 8y + \frac{7}{3} \, dy$$

$$= \left[-\frac{61y^4}{96} + \frac{17y^3}{6} - 4y^2 + \frac{7}{3}y \right]_{-2}^{2} = \frac{17 \cdot 8}{6} + \cancel{2} \frac{14}{3} - \left(-\frac{17 \cdot 8}{6} - \frac{14}{3}\right)$$

as the even terms disappear

$$= \frac{17 \cdot 16}{6} + \frac{28}{3} = \frac{17 \cdot 8 + 28}{3} = \boxed{\frac{164}{3}}$$

b) Now w.r.t. y first: $-2 \le y \le 4 - 2x$
$1 \le x \le 3$

$$J = \int_{1}^{3}\int_{-2}^{4-2x} (x-2y)^2 \, dy \, dx = \int_{1}^{3} \left[-\frac{(x-2y)^3}{6} \right]_{y=-2}^{y=4-2x} dx$$

$u = x - 2y \rightarrow$
$dy = -2dx$

$$= \int_{1}^{3} \frac{(x+4)^3 - (5x-8)^3}{6} dx = \int_{1}^{3} \frac{(x+4)^3 dx}{6} - \int_{1}^{3} \frac{(5x-8)^3}{6} dx$$

$u = x+4$, $du = dx$ $\qquad$ $v = 5x - 8$, $dv = 5\,dx$

$\cancel{= \int^{3} (x+4)^4}$

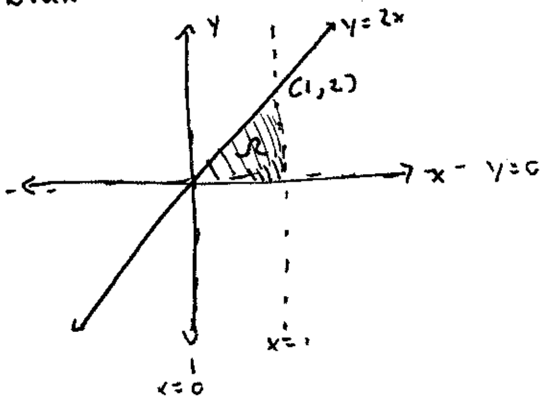
$$= \left[\frac{(x+4)^4}{24} \right]_{1}^{3} - \left[\frac{(5x-8)^4}{6 \cdot 4 \cdot 5} \right]_{1}^{3} = \frac{7^4 - 5^4}{24} - \frac{7^4 - (-3)^4}{120}$$

$$= \boxed{\frac{164}{3}}$$

✓

9. Let $K := \int_0^1 \int_0^{2x} f(x,y)\, dy\, dx$.

a) Draw a sketch of the region of integration $\Omega \subset \mathbb{R}^2$:



b) $K = \int_?^? \int_?^? f(x,y)\, dx\, dy$

We see that $y/2 \le x \le 1$, $0 \le y \le 2$ so

$$\boxed{K = \int_0^2 \left(\int_{y/2}^1 f(x,y)\, dx \right) dy}$$