Math 260

Solutions, Problem Set 11

1. $u(x,t) := \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$. Show that u satisfies the wave equation $u_{tt} = c^2 u_{xx}$ with initial position $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

$$u(x,0) = \frac{1}{2}[f(x)+f(x)] + \frac{1}{2c}\int_x^x g(s)\,ds$$

$$= f(x) + 0 = f(x). \checkmark$$

$$u_t = \frac{1}{2}[c f'(x+ct) + (-c) f'(x-ct)] + \frac{1}{2c}\frac{d}{dt}\int_{x-ct}^{x+ct} g(s)\,ds$$

$\Downarrow$ Fundamental Thm. of Calculus.

$$\frac{c}{2}[f'(x+ct) - f'(x-ct)] + \frac{1}{2c}[c g(x+ct) - (-c) g(x-ct)]$$

$$u_t(x,0) = \frac{c}{2}(f'(x) - f'(x)) + \frac{c}{2c}(g(x) + g(x))$$

$$= g(x) \checkmark$$

$$u_{tt} = \frac{c^2}{2}[f''(x+ct) + f''(x-ct)] + \frac{c}{2}(g'(x+ct) - g'(x-ct))$$

$$u_x = \frac{1}{2}[f'(x+ct) + f'(x-ct)] + \frac{1}{2c}\frac{d}{dx}\int_{x-ct}^{x+ct} g(s)\,ds$$

$\Downarrow$ FTC

$$u_x = \frac{1}{2}[f'(x+ct) + f'(x-ct)] + \frac{1}{2c}[g(x+ct) - g(x-ct)]$$

$$u_{xx} = \frac{1}{2}[f''(x+ct) + f''(x-ct) + \frac{1}{2c}(g'(x+ct) - g'(x-ct))]$$

$$\boxed{\therefore u_{tt} = c^2 u_{xx}}$$

2. Let $Q \subset \mathbb{R}^3$ be the portion of the shell $1 \le x^2+y^2+z^2 \le 9$ that is in the first octant $x \ge 0,\ y \ge 0,\ z \ge 0$.

a) Set up /evaluate the triple integral to compute the volume of Q.

Convert to polar/spherical: $1 \le \rho^2 \le 9$, $0 \le \theta \le \pi/2$, $0 \le \varphi \le \pi/2$ $\qquad dx\,dy\,dz = r^2 \sin\varphi\, dr\, d\theta\, d\varphi$

$$\text{Vol } Q = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^{3} \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\rho^3}{3}\sin\varphi\right]_1^3 d\theta\, d\varphi = \int_0^{\pi/2}\int_0^{\pi/2} \frac{26}{3}\sin\varphi\, d\theta\, d\varphi$$

$$= \frac{13\pi}{3}\int_0^{\pi/2} \sin\varphi\, d\varphi = \frac{13\pi}{3}\Big[-\cos\varphi\Big]_0^{\pi/2} = \boxed{\frac{13\pi}{3}}$$

b) Compute $I := \iiint_Q z \, dV$.

$z = \rho\cos\varphi$ so $I = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^3 \rho^3 \sin\varphi\cos\varphi\, d\rho\, d\theta\, d\varphi$

$$I = \int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\rho^4}{4}\sin\varphi\cos\varphi\right]_1^3 d\theta\, d\varphi = \int_0^{\pi/2}\int_0^{\pi/2} 20\sin\varphi\cos\varphi\, d\theta\, d\varphi = \int_0^{\pi/2} 10\pi \sin\varphi\cos\varphi\, d\varphi$$

$u = \sin\varphi$, $du = \cos\varphi$, $0 \to 0$, $\pi/2 \to 1$

$$= 10\pi\int_0^1 u\, du = \boxed{5\pi}$$

3) Find $(\bar{x}, \bar{y})$ for the semicircle $y = \sqrt{r^2 - x^2} \ge 0$ assuming uniform density.

- By symmetry, $\bar{x} = \dfrac{\iint_D x\, dx\, dy}{\iint_D dx\, dy} = \dfrac{0}{\text{nonzero}} = 0$ and

$$\iint_D y\, dx\, dy = \int_{-1}^{1}\int_0^{\sqrt{1^2-x^2}} y\, dy\, dx = \int_{-r}^{r} \frac{r^2-x^2}{2}\, dx = \left[\frac{r^2x}{2} - \frac{x^3}{6}\right]_{-r}^{r} = \frac{r^3}{2} - \frac{r^3}{6} * \left(\frac{-r^3}{2} + \frac{r^3}{6}\right)$$

$$= \frac{2r^3}{3}.$$

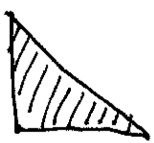
$$\iint_D dx\, dy = \frac{1}{2}(\pi r^2) = \frac{\pi r^2}{2}.$$

$$\bar{y} = \frac{\frac{2r^3}{3}}{\frac{\pi r^2}{2}} = \frac{4r}{3\pi}$$

$$\boxed{\text{CM: } \left(0, \frac{4r}{3\pi}\right)}$$

4. Find the average of e^{x+y} over the triangle w/ vertices at $(0,0)$, $(1,0)$ and $(0,1)$.

$$\left[e^{x+y}\right]_{av} = \frac{\iint_D e^{x+y}\,dx\,dy}{\iint_D dx\,dy}$$ [Marsden & Tromba, pg. 329]

D: 

$$\iint_D dx\,dy = \int_0^1\int_0^{1-y} dx\,dy = \int_0^1 1-y\,dy = \left[y-\frac{y^2}{2}\right]_0^1 = \frac{1}{2}.$$

$$\iint_D e^{x+y}\,dx\,dy = \iint_D e^x e^y\,dx\,dy = \int_0^1 e^x\,dx \int_0^{1-x} e^y\,dy$$

$$= \int_0^1 e^x\,dx \cdot (e^{1-x} - e^0) = \int_0^1 (e - e^x)\,dx = \left[ex\right]_0^1 - \left[e^x\right]_0^1$$

5, cont: $\iiint_S e^{-z}\,dx\,dy\,dz$

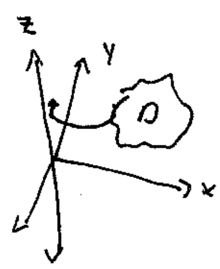
$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{-\rho\cos\varphi}\rho^2\sin\varphi\; d\theta\, d\rho\, d\varphi = 2\pi \int_0^{1}\int_0^{\pi} \rho^2 \sin\varphi\, e^{-\rho\cos\varphi}\, \cancel{d\varphi}\, \cancel{d\varphi}\, d\varphi\, d\rho$$

$u = -\rho\cos\varphi \qquad du = +\rho\sin\varphi\, d\varphi$

$$2\pi \int_0^{1} \frac{2\rho}{2}\left(e^{\rho}-e^{-\rho}\right)d\rho = 2\pi\left(1-\frac{e-2}{e}\right) = \frac{4\pi}{e}$$

so $[f_{av}] = \dfrac{\frac{4\pi}{e}}{\frac{4\pi}{3}} = \boxed{\dfrac{3}{e}}$

6. a) We're rotating around the y-axis:



So let $x = r\cos\theta$, $y = y$, $z = r\sin\theta$ to parametrize the surface of revolution W

$$Vol(W) = \iiint_W dx\,dy\,dz = \int_0^{2\pi}\iint_D r\, dr\, dy\, d\theta = 2\pi\left(\iint_D r\, dr\, dy\right)$$

(In D, the distance to the y-axis is simply $|x|$ so)

$$= 2\pi \iint_D x\, dx\, dy = 2\pi \frac{\left(\iint_D x\,dx\,dy\right)}{\iint_D dx\,dy} \cdot \iint_D dx\,dy$$

$$\boxed{Vol(W) = 2\pi \bar{x} A(D).}$$

b) rotating unit circle centered at $(3,0)$ around y-axis:

$A(D) = \pi$, $\bar{x} = 3$,

$$\boxed{\text{Volume of torus} = 2\pi \cdot 3 \cdot \pi = 6\pi^2}$$

$F = (x, y, z)$.

7. a) $\int_C F \cdot dr$, $c(t) = (t,t,t)$, $0 \le t \le 1$
$c'(t) = (1,1,1)$ so

$$= \int_0^1 \cancel{F}(c(t)) \cdot c'(t)\, dt = \int_0^1 t + t + t \, dt = \frac{3t^2}{2}\Big]_0^1 = \frac{3}{2}.$$

b) $c(t) = (\cos t, \sin t, 0)$ $\quad 0 \le t \le 2\pi$
$c'(t) = (-\sin t, \cos t, 0)$

$$\int_C F \cdot dr = \int_0^{2\pi} (\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0)\, dt$$
$$= \int_0^{2\pi} -\sin t \cos t + \cos t \sin t + 0 \, dt = \int_0^{2\pi} 0 \, dt = 0.$$

c) $c(t) = (\sin t, 0, \cos t)$ $\quad 0 \le t \le 2\pi$
$c'(t) = (\cos t, 0, -\sin t)$

$$\int_C F \cdot dr = \int_0^{2\pi} F(c(t)) \cdot c'(t)\, dt = \int_0^{2\pi} 0 \, dt = 0$$

d) $c(t) = (t^2, 3t, 2t^3)$ $\quad -1 \le t \le 2$
$c'(t) = (2t, 3, 6t^2)$

$$\int_C F \cdot dr = \int_{-1}^{2} (t^2, 3t, 2t^3) \cdot (2t, 3, 6t^2)\, dt = \int_{-1}^{2} 2t^3 + 9t + 12t^5 \, dt = \left[\frac{t^4}{2} + \frac{9t^2}{2} + \frac{12t^6}{6}\right]_{-1}^{2} = 147$$

8. a) now $F = (y, -x, z)$. Same curves from #7:

$$\int_C F \cdot dr = \int_0^1 F(c(t)) \cdot c'(t)\, dt = \int_0^1 (t, -t, t) \cdot (1,1,1) = \int_0^1 t \, dt = \frac{1}{2}.$$

b) $$\int_C F \cdot dr = \int_0^{2\pi} (\sin t, -\cos t, 0) \cdot (-\sin t, \cos t, 0) = -\int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = -2\pi$$

c) $$\int_C F \cdot dr = \int_0^{2\pi} (0, -\sin t, \cos t) \cdot (\cos t, 0, -\sin t) = -\int_0^{2\pi} \cos t \sin t \, dt = -\int_0^0 u \, du = 0.$$

$u = \sin t$
$du = \cos t \, dt$
$0 \to 0$, $2\pi \to 0$

d) $$\int_C F \cdot dr = \int_{-1}^{2} (3t, -t^2, 2t^3) \cdot (2t, 3, 6t^2)$$
$$= \int_{-1}^{2} 6t^2 - 3t^2 + 12t^5 = \int_{-1}^{2} 3t^2 + 12t^5 = \left[t^3 + 2t^6\right]_{-1}^{2} = 135$$

9. Let $c(t)$ be a path, T unit tangent vector. Then if c is between $t = a$, $t = b$:

$$\int_C T \cdot ds = \int_a^b T(c(t)) \cdot c'(t)\, dt = \int_a^b \frac{c'(t)}{\|c'(t)\|} \cdot c'(t)\, dt = \int_a^b \frac{\|c'(t)\|^2}{\|c'(t)\|}\, dt$$
$$= \int_a^b \|c'(t)\| \, dt = \text{arclength of } c \text{ (length of the path)}$$

10 a) True: If C is a vertical line segment, $\frac{dx}{dt} = 0$ for any parametrization of C. Then

$$J = \int_C V \cdot dr = \int_C p(x,y)\,dx + q(x,y)\,dy = \int_a^b p(c(t)) \cdot 0 + 0 \cdot dy_c = \int_a^b 0 = 0.$$

b) False: Let $p(x,y) = -y$ then if

$c(t) = (\cos t, \sin t)$ $0 \le t \le 2\pi$,

$c'(t) = (-\sin t, \cos t)$,

$$J = \int_C V \cdot dr = \int_0^{2\pi} (-\sin t)(-\sin t) + 0 \cdot \cos t = \int_0^{2\pi} (\sin t)^2 \geq 0$$

by positivity of $\sin t$ in $0 \le t \le 2\pi$, and because it's a continuous non-zero fn, we have $>$.

c) False: If C is a circle centered at $(0,0)$, let $p(x,y) = x$, $q(x,y) = -x$, parametrize C by $c(t) = (r\cos t, r\sin t)$ $[0 \le t \le 2\pi]$, $r \neq 0$

$$J = \int_0^{2\pi} \cancel{r}\cos t(r\sin t) + (-(\cos t)\cancel{r})(r\cos t)\,dt$$

$$J = \int_0^{2\pi} \underbrace{r^2 \cos t \sin t}_{\substack{2\pi\text{-periodic anti-derivative} \\ \text{so zero}}} - r^2\cos^2 t \; dt = -r^2 \int_0^{2\pi} \cos^2 t < 0.$$

d) Assume $J > 0$ for all parametrizations of c where

$J = \int_C V \cdot dr$. Now reverse orientation of the parametrization: call this curve ρ.

Thm 1, §7.2, Marsden & Tromba:

$$\int_\rho V \cdot dr = -\int_C V \cdot dr \quad \text{so} \quad \int_\rho V \cdot dr < 0.$$

Now as long as $p(x,y)$, $q(x,y)$ remain > 0 when reparametrizing, we have a counterexample. Let $p(x,y) = q(x,y) = c$ for $c > 0$ constant. ∎

11. x^3 is an odd function in $(+x)$:

$f(x) = x^3$, $f(-x) = (-x)^3 = (-1)^3 x^3 = -x^3 = -f(x)$ so

as Ω is symmetrical over the ~~y-axis,~~ yz-plane, split Ω into two symmetrical parts, Ω^+ and Ω^-. $\left(\begin{array}{l}\Omega^+ : \Omega : x > 0, \\ \Omega^- : \Omega : x \le 0\end{array}\right)$

$$\iiint_{\Omega^+} x^3\,dV + \iiint_{\Omega^-} x^3\,dV = \iiint_{\Omega} x^3\,dV \text{ but}$$

$$\iiint_{\Omega^+} x^3\,dV = -\iiint_{\Omega^-} x^3\,dV \quad \text{as } x^3 \text{ is odd.}$$

$$\therefore \iiint_\Omega x^3\,dV = 0$$

12 a) Note that if $F = \nabla u(x,y,z)$,

$$\frac{dF_1}{dy} = \frac{d^2 u}{dy\,dx} = \frac{d^2 u}{dx\,dy} = \frac{d}{dx}\left(\frac{du}{dy}\right) = \frac{dF_2}{dx}$$ and similarly for all

(assume u is C^2)

b) $F = (2x, z, 2y)$

$\frac{dF_2}{dz} = 1 \neq 2 = \frac{dF_3}{dy}$ so by a, F is not conservative.

c) If $F = (2xy^3, 3x^2y^2)$,

$\int F_x\, dx = x^2y^3 + g(y)$, $\int F_y\, dy = x^2y^3 + h(x)$

so $g(y) = h(x) = c$ constant,

$F = \nabla u$ where $u(x,y) = x^2y^3 + c \quad \forall c \in \mathbb{R}$.

d) $F: (2xyz, x^2z, x^2y)$

$\int F_x\, dx = x^2yz + g(y,z)$, $\int F_y\, dy = x^2yz + h(x,z)$, $\int F_z\, dz = x^2yz + \ell(x,y)$

so $g(y,z) = h(x,z) = \ell(x,y) = c$ constant gives us a consistent potential of $x^2yz + c$ for F.

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13 [Should read as] $\displaystyle\int_C 2xyz\, dx + x^2z\, dy + x^2y\, dz$

C: any oriented simple curve connecting $(1,1,1)$ to $(1,2,4)$.

From 12 d, our vector field is conservative, so

$\begin{bmatrix}\text{Thm 3,} \\ \S 7.2\end{bmatrix}$ $\displaystyle\int_C (2xyz, x^2z, x^2y)\cdot ds = u(1,2,4) - u(1,1,1)$ where

$u = x^2yz + c \quad ; c \in \mathbb{R}$.

$u(1,2,4) - u(1,1,1) =$

$1^2\cdot 2\cdot 4 + c - (1^2\cdot 1\cdot 1 + c) = 8 - 1 = 7$.

14. $\nabla f(x,y,z) = 2xyz\, e^{x^2}\,\hat{i} + z e^{x^2}\,\hat{j} + y e^{x^2}\,\hat{k}$, $f(0,0,0) = 5$,

$f(1,1,2) = ?$

$\int \nabla f_x\, dx = yze^{x^2} + g(y,z)$,

$\int \nabla f_y\, dy = yze^{x^2} + h(x,z)$,

$\int \nabla f_z\, dz = yze^{x^2} + \ell(x,y)$

$g(y,z) \neq h(x,z) = \ell(x,y) =$ constant but if they $= c$,

$f = yze^{x^2} + c$ and

$f(0,0,0) = 0\cdot 0\cdot e^0 + c = 5$, $c = 5$ so

so $f(1,1,2) = 1\cdot 2\cdot e^{1^2} + 5 = \boxed{2e+5}$