Math 260 **Exam 3** Jerry L. Kazdan
April 12, 2012 12:00 – 1:20

DIRECTIONS This exam has two parts. PART A has 4 short answer questions (10 points each, so 40 points) while PART B has 4 traditional problems (15 points each, so 60 points). Total: 100 points. *Neatness counts.*

Closed book, no calculators, computers, ipods, cell phones, etc – but you may use one $3'' \times 5''$ card with notes on both sides.

Part A: Four short answer questions (10 points each, so 40 points).

A–1. Let $f(x) := \int_0^x \left(\int_0^t g(s)\,ds \right) dt$ for $x \ge 0$. Rewrite this as an iterated integral with the order of integration reversed, so one first integrates with respect to t.

SOLUTION: In the st-plane the region of integration is the triangle with vertices at $(0,0)$, $(0,\,x)$, and $(x,\,x)$. Draw a sketch! If you integrate first with respect to t, you get

$$J = \int_0^x \left(\int_s^x g(s)\,dt \right) ds = \int_0^x (x-s)g(s)\,ds.$$

REMARK: If you solve $f'' = g$ with initial conditions $f(0) = 0$ and $f'(0) = 0$ by integrating twice, you get the first formila for J. Interchanging the order of integration gives the second, which is a bit simpler. You can also get the second version from the first by an integration by parts.

For the next 3 problems, $\gamma(t) = (x(t),\,y(t))$, $a \le t \le b$, is a smooth curve in the plane and we consider the line integral $J := \int_\gamma p(x,y)\,dx + q(x,y)\,dy$. Give a proof or counterexample for each of the following.

A–2. If $\gamma(t)$ is a *horizontal* line segment and $p(x,y) = 0$ on this segment, then $J = 0$.

SOLUTION: Parametrize the horizontal line segment, say $a \le x \le b$ as $\gamma(t) = (t,\,c)$, where $a \le t \le b$. Then $dx/dt = 1$ and $dy/dt = 0$. Thus

$$J := \int_\gamma \left[p(x,y)\,\frac{dx}{dt} + q(x,y)\frac{dy}{dt} \right] dt = \int_a^b \left[0\,\frac{dx}{dt} + q(x,y)0 \right] dt = 0.$$

A–3. If $\gamma(t)$ is a *vertical* line segment and $p(x,y) = 0$ on this segment, then $J = 0$.

SOLUTION: If $\alpha \le y \le \delta$ and $x = c$, write the curve as $\gamma(t) = (c,\,t)$, with , $\alpha \le t \le \delta$. Then $dx/dt = 0$ and

$$J := \int_\gamma \left[p(x,y)\,\frac{dx}{dt} + q(x,y)\frac{dy}{dt} \right] dt = \int_\alpha^\beta [0 + q(c,t)t]\,dt.$$

It is easy to pick q so that this is not zero – and we get a counterexample. For instance use $q(x,y) \equiv 1$.

A–4. If $p(x,y) \geq 0$ and $q(x,y) \geq 0$ on γ, and if in defining γ we know that $dx/dt > 0$ and $dy/dt > 0$, then $J \geq 0$.

SOLUTION: Just as above,

$$J := \int_\gamma \left[p(x,y)\,\frac{dx}{dt} + q(x,y)\frac{dy}{dt} \right]\,dt$$

but now all the terms in the integrand are non-negative. Hence $J \geq 0$.

Part B: Four traditional problems (15 points each, so 60 points).

B–1. Let $\mathbf{F} = y\mathbf{i} + (3+2x)\mathbf{j} + 2\mathbf{k}$, and $\gamma(t)$ be the straight line from $(0,0,0)$ to $(1,2,-3)$. Compute $\displaystyle\int_\gamma \mathbf{F}\cdot d\mathbf{s}$.

SOLUTION: First note

$$\int_\gamma \mathbf{F}\cdot d\mathbf{s} = \int_\gamma \mathbf{F}(\gamma(t))\cdot\frac{d\gamma}{dt}\,dt.$$

Parametrize γ as $\gamma(t) = (t,\,2t,\,-3t)$, for $0 \leq t \leq 1$. Then $\gamma'(t) = (1,\,2,\,-3)$ and on γ we have

$$\begin{aligned} \mathbf{F}(\gamma(t))\cdot\frac{d\gamma}{dt} &= (2t,\,3+2t,\,2)\cdot(1,\,2,\,-3) \\ &= 2t + 2(3+2t) - 6 = 6t. \end{aligned}$$

Therefore

$$\int_\gamma \mathbf{F}\cdot d\mathbf{s} = \int_0^1 6t\,dt = 3.$$

B–2. Let $G(x) := \displaystyle\int_{a(x)}^{b(x)} f(t)\,dt$, where $a(x)$ and $b(x)$ are smooth functions with $a(x) < b(x)$, and $f(x)$ is a continuous function. Compute $dG(x)/dx$.

SOLUTION: For real numbers $p < q$ let $H(p,q) := \int_p^q f(t)\,dt$, Then $G(x) = H(a(x),\,b(x))$ so by the chain rule with $p = a(x)$, and $q = b(x)$

$$\begin{aligned} \frac{dG}{dx} &= \frac{\partial H}{\partial p}\frac{dp}{dx} + \frac{\partial H}{\partial q}\frac{dq}{dx} \\ &= -\,f(a(x))\frac{da}{dx} + f(b(x))\frac{db}{dx} \\ &= f(b(x))\frac{db}{dx} - f(a(x))\frac{da}{dx}. \end{aligned}$$

B–3. Compute $J := \displaystyle\iint_{\mathbb{R}^2} \frac{dx\,dy}{[4+(x-y)^2+(x+2y)^2]^2}$

Solution: Since we recognize that $\displaystyle\iint_{\mathbb{R}^2} \frac{du\,dv}{[4+u^2+v^2]^2}$ is probably doable by using polar coordinates, it is natural to try the preliminary substitution $u = x - y, \quad v = x + 2y$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = 3 \qquad \text{so} \qquad du\,dv = 3\,dx\,dy.$$

Thus,

$$J = \iint_{\mathbb{R}^2} \frac{1}{[4+u^2+v^2]^2} \frac{du\,dv}{3}.$$

Now we change to polar coordinates, $u = r\cos\theta, \quad v = r\sin\theta$ and find

$$J = \frac{1}{3}\int_0^{2\pi} \left(\int_0^\infty \frac{1}{[4+r^2]^2} r\,dr \right) d\theta = \frac{\pi}{12}$$

B–4. Let the surface $S \subset \mathbb{R}^3$ be the graph of $z = g(x,y)$ for $(x,\,y)$ in a region D in the xy-plane.

a) Using the parameters $x = u, \; y = v, \; z = g(u,v)$, derive the formula

$$\text{Area}\,(S) = \iint_D \sqrt{1 + \|\nabla g\|^2}\,dx\,dy.$$

Solution: We begin with the formula for the area of a surface defined parametrically over a region Ω in the (u,v) parameter space:

$$\text{Area}\,(S) = \iint_\Omega \|T_u \times T_v\|\,du\,dv,$$

where $T_u = (x_u,\, y_u)$ and $T_v = (x_v,\, y_v)$ are tangent vectors on the surface. In this special case, $T_u = (1,\, 0,\, g_u(u,v)$ and $T_v = (0,\, 1,\, g_v(u,v)$. Then by a routine computation, $T_u \times T_v = (-g_u,\, -g_v, 1)$ so $\|T_u \times T_v\| = \sqrt{1 + g_u^2 + g_v^2}$. Since $u = x$ and $v = y$, we conclude

$$\text{Area}\,(S) = \iint_\Omega \sqrt{1 + g_u^2 + g_v^2}\,du\,dv = \iint_D \sqrt{1 + \|\nabla g\|^2}\,dx\,dy.$$

b) Apply this to compute the surface area of the part of the plane $x + 2y + z = 2$ in the first octant $x \ge 0, \quad y \ge 0, \quad z \ge 0$.

Solution: Here $z = g(x,y) = 2 - x - 2y$ so $\nabla g = (-1,\, -2)$ and $\|\nabla g\|^2 = 5$.
The plane passes through the three points $(2,\, 0,\, 0)$, $(0,\, 1,\, 0)$, $(0,\, 0,\, 2)$ and intersects the horizontal plane, $z = 0$, in the line $x + 2y = 2$. Thus the region D is the triangle in the xy-plane bounded by the x-axis, the y-axis, and the line $x + 2y = 2$. This gives

$$\text{Area}\,(S) == \int_0^1 \left(\int_0^{2-2y} \sqrt{1+5}\,dx \right) dy = \sqrt{6}\int_0^1 (2-2y)\,dy = \sqrt{6}$$