Math 260 | Exam 2 | Jerry L. Kazdan
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DIRECTIONS This exam has two parts. PART A has 6 short answer questions (7 points each, so 42 points) whilePART B has 4 traditional problems (15 points each, so 60 points). Total: 102 points. *Neatness counts.*

Closed book, no calculators, computers, ipods, cell phomes, etc – but you may use one $3'' \times 5''$ card with notes on both sides.

PART A: Six short answer questions (7 points each, so 42 points).

1. Find a 3×3 symmetric matrix A with the property that

$$\langle X,\, AX\rangle = -x_1^2 + 6x_1x_2 - x_1x_3 + 2x_2x_3 + 3x_2^2$$

for all $X = (x_1,\, x_2,\, x_3) \in \mathbb{R}^3$.

SOLUTION: $A := \begin{pmatrix} -1 & 3 & -\frac{1}{2} \\ 3 & 3 & 1 \\ -\frac{1}{2} & 1 & 0 \end{pmatrix}$

2. Under what conditions on the constants a, b, c, and d is the following matrix A positive definite?

$$A := \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

SOLUTION: Let $X := (x_1, x_2, x_3, x_4)$ Then

$$\langle X,\, AX\rangle = ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2 > 0 \quad \text{for all} \quad X \neq 0$$

if and only if $a > 0$, $b > 0$, $c > 0$, and $d > 0$.

3. Let B be an anti-symmetric $n\times n$ real matrix, so $B^* = -B$. Show that $\langle V,\, BV\rangle = 0$ for all $V \in \mathbb{R}^n$.

SOLUTION: $\langle V,\, BV\rangle = \langle B^*V,\, V\rangle = -\langle BV,\, V\rangle = -\langle V,\, BV\rangle$. Thus $2\langle V,\, BV\rangle = 0$ and hence $\langle V,\, BV\rangle = 0$.

4. Find the arc length of the segment of the helix $X(t) := (\cos 3t, 1-4t, \sin 3t)$, for $0 \le t \le \pi$.

SOLUTION: Arc length $= \int_0^\pi \|X'(t)\|\, dt$. But $X'(t) = (-3\sin 3t, -4, 3\cos 3t)$ so $\|X'(t)\|^2 = 9\sin^2 3t + 16 + 9\cos^2 3t = 25$. Thus

$$\text{Arc Length} = \int_0^\pi 5\, dt = 5\pi.$$

5. Find some function $u(x,y)$ that satisfies $\dfrac{\partial^2 u}{\partial x \partial y} = 4\cos(x+2y) - 2xy$.

SOLUTION: First integrate with respect to x to find $u_y(x,y) = 4\sin(x+2y) - x^2 y + g(x)$, where the "constant" of integration, $g(y)$, is any function of y. Now integrate with respect to y:

$$\begin{aligned} u(x,y) &= -2\cos(x+2y) - \frac{x^2y^2}{2} + \int g(y)\,dy + h(x) \\ &= -2\cos(x+2y) - \frac{x^2y^2}{2} + f(y) + h(x), \end{aligned}$$

where $f(y)$ and $h(x)$ are any functions of their variables. Since the problem only asked for "some function", we can choose $f(y) = 0$ and $h(x) = 0$.

Note that we could have first integrated with respect to y.

6. Let $v(s)$ be a smooth function of the real variable s and let $u(x,t) := v(x+3t)$. Show that u satisfies the homogeneous partial differential equation $u_t - 3u_x = 0$.

SOLUTION: Let v' denote the derivative of v. Then by the chain rule $u_x(x,t) = v'(x+3t)\cdot 1$ and $u_t(x,t) = v'(x+3t)\cdot 3$. Thus $3u_x(x,t) = u_t(x,t)$ as desired.

PART B: Four traditional problems (15 points each, so 60 points).

B–1. In an experiment, at time t you measure the value of a quantity R and obtain the data:

t	-1	0	1	2
R	-1	1	1	-3

Based on other information, you believe this data should fit a curve of the form $R = a + bt^2$.

a) Write the (over-determined) system of linear equations you would ideally like to solve for the unknown coefficients a and b.

SOLUTION:

$$\begin{aligned} a + b &= -1 \\ a + 0 &= 1 \\ a + b &= 1 \\ a + 4b &= -3 \end{aligned}$$

b) Use the method of least squares to find the *normal equations* for the coefficients a and b.

SOLUTION: Let $A := \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{pmatrix}$, $\quad V := \begin{pmatrix} a \\ b \end{pmatrix}$, $\quad$ and $\quad w := \begin{pmatrix} -1 \\ 1 \\ 1 \\ -3 \end{pmatrix}$.

The *normal equations* are $A^*AV = A^*w$, that is,

$$\begin{pmatrix} 4 & 6 \\ 6 & 18 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ -12 \end{pmatrix}.$$

c) Solve the normal equations to find the coefficients a and b.

SOLUTION: These equations are

$$\begin{aligned} 4a + 6b &= -2 \\ 6a + 18b &= -12 \end{aligned} \qquad \text{that is,} \qquad \begin{aligned} 2a + 3b &= -1 \\ a + 3b &= -2 \end{aligned}$$

The solution is $a = 1$, $b = -1$. Thus the equation of the least squares curve is $R = 1 - t^2$.

B–2. Find and classify all the critical points of $f(x, y, z) := x^3 - 3x + y^2 + z^2$.

SOLUTION: The critical points are where $\nabla f = 0$, that is, $0 = f_x = 3x^2 - 3x$, $0 = f_y = 2y$, and $0 = f_z = 2z$. Thus $x = \pm 1$, $y = 0$, and $z = 0$. The critical points are thus $P_1 := (1, 0, 0)$, $P_2 = (-1, 0, 0)$.

To classsify these we use the second derivative ("Hessian") matrix

$$f''(x, y, z) = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In particular,

$$f''(P_1) = f''(1, 1, 1) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad f''(P_2) = f''(-1, 1, 1) = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since $f''(P_1)$ is positive definite, f has a local min at P_1. However, two of the diagonal elements of $f''(P_2)$ have opposite sign so it is indefinite. Hence f has a saddle point at P_2.

B–3. For a certain rod of length π, the temperature $u(x, t)$ at the point x at time t satisfies the heat equation $u_t = u_{xx}$. Find *all* solutions of the special form

$$u(x, t) = w(x)T(t) \qquad \text{for} \qquad 0 \le x \le \pi$$

that satisfy the boundary conditions $u(0, t) = 0$ and $u(\pi, t) = 0$ for all $t \ge 0$. [We seek the *non-trivial solutions*, that is, other than the important but uninteresting solution $u(x, t) \equiv 0$.]

SOLUTION: Note that the boundary conditions imply $0 = u(0,t) = w(0)T(t)$ and $0 = u(\pi,t) = w(\pi)T(t)$ for all $t \ge 0$. Consequently $w(0) = 0$ and $w(\pi) = 0$.

Substituting $u(x,t) = w(x)T(t)$ into the heat equation and separating variables we get

$$\frac{1}{T(t)}\frac{dT(t)}{dt} = \frac{1}{w(x)}\frac{d^2w(x)}{dx^2} = \alpha,$$

where α is a constant. Thus

$$w'' = \alpha w \qquad \text{and} \qquad \frac{dT}{dt} = \alpha T.$$

We claim that $\alpha < 0$ (this is a key step). To show this, multiply both sides of $w''(x) = \alpha w(x)$ by $w(x)$ and integrate over the rod. Then integrate by parts and use the boundary conditions $w(0) = w(\pi) = 0$ to get

$$\alpha \int_0^\pi w(x)^2\,dx = \int_0^\pi w(x)w''(x)\,dx = -\int_0^\pi w'(x)^2\,dx \le 0.$$

This already implies that $\alpha \le 0$. However, if $\alpha = 0$ then $w'(x)^2 = 0$ so $w(x) = \text{constant}$. But $w(0) = 0$. Thus $w(x) \equiv 0$. This gives the trivial solution $u(x,t) \equiv 0$ which we discard. Consequently $\alpha < 0$ so we write $\alpha = -\lambda^2$.

Thus $w''(x) + \lambda^2 w(x) = 0$ whose general solution is $w(x) = A\cos\lambda x + B\sin\lambda x$. The boundary condition $w(0) = 0$ implies $A = 0$ while the boundary condition $w(\pi) = 0$ implies $B\sin\lambda\pi = 0$. We exclude the possibility that $B = 0$ since this gives us the trivial solution $u(x,t) \equiv 0$. Consequently, $\sin\lambda\pi = 0$, so $\lambda = k = 1, 2, \ldots$ and $\alpha = -k^2$ so the solution of $dT/dt = \alpha T$ is $T(t) = Ce^{-k^2 t}$.

Collecting our results we have the special solutions

$$u_k(x,t) = C_k \sin(kx)e^{-k^2 t}, \qquad k = 1, 2, \ldots .$$

B–4. Say the equation $f(X) := f(x,y,z) = 0$ implicitly defines a smooth surface in $\mathbb{R}^3$ (an example is the sphere $x^2 + y^2 + z^2 - 4 = 0$). Let $P \in \mathbb{R}^3$ be a point *not* on this surface. Assume Q is a point on the surface that is closest to P. Show that the vector from P to Q is orthogonal to the tangent plane to the surface at Q.

[SUGGESTION: Let $X(t)$ be a smooth curve in the surface with $X(0) = Q$. Then Q is the point on the curve that is closest to P.]

SOLUTION: Using the curve $X(t)$, let $h(t) := \|X(t) - P\|^2$. Since $h(t)$ is minimized at $t = 0$, then $h'(0) = 0$. Now $h(t) = \langle X(t) - P,\, X(t) - P\rangle$ so

$$h'(t) = \langle X'(t),\, X(t) - P\rangle + \langle X(t) - P,\, X'(t)\rangle = 2\langle X'(t),\, X(t) - P\rangle.$$

At $t = 0$ this gives

$$0 = \langle X'(0),\, Q - P\rangle,$$

that is, the vector $Q - P$ is perpendicular to the vector $X'(0)$ that is tangent to the surface at Q. Since ths is true for any tangent vector at Q, the vector $Q - P$ is perpendiculat to the whole tangent plane at Q.

REMARK: You can also prove this result using Lagrange Multipliers.

NOTE: A common error was to take the derivative of $\|Q - P\|^2$. This fails because P and Q are specified points so the derivative of the constant $\|Q - P\|^2$ is zero for trivial reasons. It gives no information.