Math 260 | **Exam 2** | Jerry L. Kazdan

March 13, 2012 | 12:00 – 1:20

DIRECTIONS This exam has two parts. PART A has 6 short answer questions (7 points each, so 42 points) whilePART B has 4 traditional problems (15 points each, so 60 points). Total: 102 points. *Neatness counts.*

Closed book, no calculators, computers, ipods, cell phomes, etc – but you may use one $3'' \times 5''$ card with notes on both sides.

PART A: six short answer questions (7 points each, so 42 points).

1. Find a 3×3 symmetric matrix A with the property that
$$\langle X,\, AX \rangle = -x_1^2 + 6x_1x_2 - x_1x_3 + 2x_2x_3 + 3x_2^2$$
for all $X = (x_1,\, x_2,\, x_3) \in \mathbb{R}^3$.

2. Under what conditions on the constants a, b, c, and d is the following matrix A positive definite?
$$A := \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

3. Let B be an anti-symmetric $n \times n$ real matrix, so $B^* = -B$. Show that $\langle V,\, BV \rangle = 0$ for all $V \in \mathbb{R}^n$.

4. Find the arc length of the segment of the helix $X(t) := (\cos 3t, 1 - 4t, \sin 3t)$, for $0 \le t \le \pi$.

5. Find some function $u(x, y)$ that satisfies $\dfrac{\partial^2 u}{\partial x \partial y} = 4\cos(x + 2y) - 2xy$.

6. Let $v(s)$ be a smooth function of the real variable s and let $u(x, t) := v(x + 3t)$. Show that u satisfies the homogeneous partial differential equation $u_t - 3u_x = 0$.

[PART B IS ON THE NEXT PAGE]

PART B: four traditional problems (15 points each, so 60 points).

B–1. In an experiment, at time t you measure the value of a quantity R and obtain:

t	-1	0	1	2
R	-1	1	1	-3

Based on other information, you believe the data should fit a curve of the form $R = a + bt^2$.

a) Write the (over-determined) system of linear equations you would ideally like to solve for the unknown coefficients a and b.

b) Use the method of least squares to find the *normal equations* for the coefficients a and b.

c) Solve the normal equations to find the coefficients a and b.

B–2. Find and classify all the critical points of $f(x, y, z) := x^3 - 3x + y^2 + z^2$.

B–3. For a certain rod of length π, the temperature $u(x, t)$ at the point x at time t satisfies the heat equation $u_t = u_{xx}$. Find *all* solutions of the special form

$$u(x, t) = w(x)T(t) \qquad \text{for} \qquad 0 \le x \le \pi$$

that satisfy the boundary conditions $u(0, t) = 0$ and $u(\pi, t) = 0$ for all $t \ge 0$.

B–4. Say the equation $f(X) := f(x, y, z) = 0$ implicitly defines a smooth surface in $\mathbb{R}^3$ (an example is the sphere $x^2 + y^2 + z^2 - 4 = 0$). Let $P \in \mathbb{R}^3$ be a point *not* on this surface. Assume Q is a point on the surface that is closest to P. Show that the vector from P to Q is orthogonal to the tangent plane to the surface at Q.

[SUGGESTION: Let $X(t)$ be a smooth curve in the surface with $X(0) = Q$. Then Q is the point on the curve that is closest to P.]