

Remark on Problem Set 9 #1

Problem 1 [See Section 7.3 in Haberman] Solve the wave equation $u_{tt} = c^2 u_{xx}$ in the square $\Omega = \{0 < x < \pi, 0 < y < \pi\}$ in the plane with $\nabla u \cdot N = 0$ on the boundary and initial conditions $u(x, y, 0) = 0$, $u_t(x, y, 0) = \sin^2 x$.

Remarks: Say one is solving the wave equation

$$u_{tt} = c^2 \Delta u$$

in a bounded region $\Omega \subset \mathbb{R}^2$ with *boundary condition* $\nabla u \cdot N = 0$ on the boundary and *initial conditions* $u(x, y, 0) = f(x, y)$, $u_t(x, y, 0) = g(x, y)$.

Make the preliminary separation of variables $u(x, y, t) = v(x, y)T(t)$.

Then

$$\frac{T''(t)}{c^2 T} = \frac{\Delta v}{v} = -\lambda$$

for some constant λ . Thus

$$T'' + c^2 \lambda T = 0 \quad \text{and} \quad \Delta v + \lambda v = 0. \quad (1)$$

Using the boundary condition, we find that $\lambda \geq 0$. Here the possibility $\lambda = 0$ does arise, with corresponding eigenfunction $v(x, y) = \text{constant}$. For instance, we can let this $v(x, y) = 1$. It is customary to number this special eigenvalue as $\lambda_0 = 0$ and number the positive eigenvalues as $\lambda_1 \leq \lambda_2 \leq \dots$. Then $v_0(x, y) = 1$ and $\Delta v_n + \lambda_n v_n = 0$, $n = 0, 1, 2, \dots$. The solution of the ODE (1) for $T(t)$ then fall into two cases:

$$\begin{aligned} n = 0 : & \quad T_0(t) = a_0 + b_0 t \\ n \geq 1 : & \quad T_n(t) = a_n \cos(c\sqrt{\lambda_n}t) + b_n \sin(c\sqrt{\lambda_n}t). \end{aligned}$$

Thus, using $v_0(x, t) = 1$, the general solution of the wave equation with these boundary conditions is

$$\begin{aligned} u(x, y, t) &= \sum_{n=0}^{\infty} T_n(t)v_n(x, y) = T_0(t)v_0(x, y) + \sum_{n=1}^{\infty} T_n(t)v_n(x, y) \\ &= a_0 + b_0t + \sum_{n=1}^{\infty} \left[a_n \cos(c\sqrt{\lambda_n}t) + b_n \sin(\sqrt{\lambda_n}t) \right] v_n(x, y) \end{aligned}$$

The coefficients a_n and b_n are now computed using the initial position, $u(x, y, 0)$ and initial velocity, $u_t(x, y, 0)$.

In this particular problem, you should have found

$$v_{k\ell}(x, y) = \cos kx \cos \ell y \quad \text{for } k, \ell = 0, 1, 2, \dots \quad \text{and} \quad \lambda_{k\ell} = k^2 + \ell^2.$$

and the above formula for $u(x, y, t)$ becomes

$$\begin{aligned} u(x, y, t) &= a_{00} + b_{00}t \\ &\quad + \sum_{k, \ell=1}^{\infty} \left[a_{k\ell} \cos(c\sqrt{\lambda_{k\ell}}t) + b_{k\ell} \sin(c\sqrt{\lambda_{k\ell}}t) \right] \cos kx \cos \ell y \end{aligned}$$

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