Math 241, Spring 2015 **Jerry L. Kazdan**

Problem Set 8

DUE: Thurs. April 2 in class. [Late papers will be accepted until 1:00 PM Friday.]

This week: Please read Chapter 7, Sections 7.1 – 7.9 in the Haberman text.

1. p. 280 #7.3.5
2. p. 287 #7.4.4
3. p. 290 #7.5.1
4. p. 290 #7.5.4
5. p. 291 #7.5.8(a)
6. p. 295 #7.6.3 (assume $p > 0$ and $\sigma > 0$).
7. p. 295 # 7.6.4
8. p. 308 #7.7.2(a) (in a disk of radius 1)
9. p. 308 #7.7.7
10. Say $u(x)$ satisfies $u'' + xu' + xu = 0$ Find a change of variable $u(x) = \phi(x)v(x)$ so v satisfies an equation with no v' term (we used this in class on Thursday).
11. a) Let B be the ball $\{\rho^2 = x^2 + y^2 + z^2 < a^2\}$ in $\mathbb{R}^3$. Compute all the *radial* eigenfunctions $w(\rho)$ of $-\Delta$ with Dirichlet boundary conditions $w(a) = 0$. Thus, you are solving $-[w_{\rho\rho} + \frac{2}{\rho} w_\rho] = \lambda w$. [SUGGESTION: the substitution $v(\rho) = \rho w(\rho)$ is useful. Note it implies $v(0) = 0$.]

    b) Compute the corresponding eigenvalues

    c) Use this to solve the heat equation $u_t = \Delta u$ in B with $u(a) = 0$ on the boundary in the special case where the initial temperature, $u(x, 0) = \varphi(\rho)$ depends only on ρ. Your solution will be an infinite series. Please include a formula for finding the coefficients.

Bonus Problem

[Please give this directly to Professor Kazdan]

B-1 On the interval $\alpha \le x \le \beta$ let $u(x)$ and $v(x)$ be solutions of the equations

$$Lu := u'' + a(x)u = 0 \qquad \text{and} \qquad Mv := v'' + b(x)v = 0,$$

where $a(x)$ and $b(x)$ are continuous functions.

a) Show that

$$0 = \int_\alpha^\beta (vLu - uMv)\,dx == \left[vu' - uv'\right|_\alpha^\beta - \int_\alpha^\beta (b-a)uv\,dx$$

b) Suppose that α and β are consecutive zeroes of u and that $u(x) > 0$ in the interval $\alpha < x < \beta$. If $a(x) \le b(x)$ in this interval, show that $v(x)$ must be zero somewhere in this interval by using that $u'(\alpha) > 0$ and $u'(\beta) < 0$ and that there is a contradiction unless v is zero somewhere in this interval. This is the *Sturm oscillation theorem*.

c) In the special case where $a(x) = b(x)$ and $u(x)$ and $v(x)$ are linearly independent solutions of the same equation, conclude that between any two zeroes of u there is a zero of v, and vice versa. Thus the zeroes interlace. A special case of this is the interlacing of the zeroes of $\sin x$ and $\cos x$.

d) If $b(x) \ge c^2 > 0$ for some constant c, show that $v(x)$ must have infinitely many zeroes by comparing v with a collation of $u'' + c^2 u = 0$.

e) Show that every solution of $v'' + (1 - \frac{1}{x^2}v = 0$ must have infinitely many zeroes .

[Last revised: April 1, 2015]