Math 241, Spring 2015 **Jerry L. Kazdan**

Problem Set 6

DUE: Thurs. March 5 in class. [Late papers will be accepted until 1:00 PM Friday.]

This week. Please read Chapter 5, Sections 5.1 – 5.7 in the Haberman text.

1. Let $f(x) = x^2$, $0 \le x \le \pi$.

   a) Extended f to all real x as an even function that is 2π periodic. For which values of x does its Fourier series converge point wise to f? At the point(s) where it does not converge, to what does it converge?

   b) Extended f to all real x as an odd function that is 2π periodic. For which values of x does its Fourier series converge point wise to f? At the point(s) where it does not converge, to what does it converge?

2. Let f be a 2π periodic function whose first derivative is continuous. How are the Fourier coefficients of f and f' related? [SUGGETION: Integrate by parts.]

3. [Related to p. 129, 3.6.2] Let f have the complex Fourier series $f(x)\ \sum_{-\infty}^{\infty} c_n e^{inx}$.

   a) If f is real-valued, show that $c_{-n} = \overline{c_n}$.

   b) If $c_{-n} = \overline{c_n}$, show that f is real-valued.

4. Let $f(x)$ be a 2π periodic real function with (complex) Fourier series $f(x) = \sum_{-\infty}^{\infty} a_n e^{inx}$ and let α be a real constant. We seek a 2π periodic solution $u(x)$ of

   $$u'' + \alpha u = f(x)$$

   as a Fourier series $u(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$.

   a) Formally find the desired coefficients c_n in terms of the Fourier coefficients a_n of f.

   b) Are there any values of α for which there are difficulties? Explain. In particular, if $\alpha = k^2$ for some integer k, show that a necessary condition for a solution to exist is that

   $$\int_{-\pi}^{\pi} f(x) \cos kx\, dx = 0 \qquad \text{and} \qquad \int_{-\pi}^{\pi} f(x) \sin kx\, dx = 0.$$

5. a) Find the 2π periodic Fourier series for $f(x) = \begin{cases} 0 & \text{for } -\pi \le x \le 0 \\ x & \text{for } 0 \le x < \pi \end{cases}$.

   b) Use the boxed theorem p. 123 (top) to find the Fourier series for $f(x) = \begin{cases} 0 & \text{for } -\pi \le x \le 0 \\ x^2 & \text{for } 0 \le x < \pi \end{cases}$.

Remark: As is pointed out near the bottom of p. 124 (#2), since the Fourier series for this $f(x)$ has $a_0 \neq 0$, its integral involves $a_0 x$ so you will also need the 2π periodic Fourier (sine) series for x:

$$x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots + (-1)^{n+1}\frac{\sin nx}{n} + \cdots\right].$$

6. Use d'Alembert's formula (p. 508, eq 11.2.36) to solve the wave equation $u_{tt} = c^2 u_{xx}$ for $-\infty < x < \infty$ with the following initial conditions:

   a) $u(x,0) = \dfrac{1}{1+x^2}$, and $u_t(x,0) = 0$.

   b) $u(x,0) = 0$ and $u_t(x,0) = 4$.

7. d'Alembert's formula (p. 508, eq 11.2.36) gives a formula for the solution of $u_{tt} = c^2 u_{xx}$ for $-\infty < x < \infty$ with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

   The goal of this problem is to show how to modify d'Alembert's formula to solve the initial value problem for the wave equation on the semi-infinite interval $x \geq 0$ given $u(x,0) = F(x)$ and $u_t(x,0) = G(x)$, where F and G are only defined for $x \geq 0$ but you also have the additional boundary condition $u(0,t) = 0$ for all $t \geq 0$. You may assume the compatibility conditions $F(0) = 0$ and $G(0) = 0$.

   a) In the d'Alembert formula, if both $f(x)$ and $g(x)$ are odd smooth functions, directly from this formula show that the solution, $u(x,t)$ is an odd function of x. Consequence: $u(0,t) = 0$ for all $t \geq 0$

   b) For the semi-infinite interval problem where $x \geq 0$, $F(x)$ and $G(x)$ are defined only for $x > 0$. Let f_{odd} and g_{odd} be their extensions to all $x \in \mathbb{R}$ as odd functions. Then let $u(x,t)$ be the solution that d'Alembert's formula gives for the solution of the wave equation for all real x.

      Show that if you only use this $u(x,t)$ for $x \geq 0$, it is a solution of the wave equation for $x > 0$ with $u(x,0) = F(x)$,$u_t(x,0) = G(x)$ with the end at $x = 0$ fixed: $u(0,t) = 0$ for all $t > 0$. [For u to be smooth enough, assume that F and G and their odd extensions are smooth.]

8. Say $u_{tt} = 9u_{xx}$ for all $-\infty < x < \infty$ with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Find the largest interval $J = \{a \leq x \leq b\}$ where modifying f and/or g *inside* this interval can change the value of $u(6,5)$.

[Last revised: March 4, 2015]