Math 241, Spring 2015 **Jerry L. Kazdan**

Problem Set 5

DUE: Thurs. Feb. 26 in class. [Late papers will be accepted until 1:00 PM Friday.]

This week. Please read all of Chapter 3, and Chapter 4, Sec. 4.1 - 4.5 in the Haberman text.

1. Suppose that u satisfies the heat equation in a rod ($0 < x < L$) with the various boundary conditions on the left below. In the list below, match the boundary condition on the left with the appropriate reference temperature u_0 on the right so that $v = u - u_0$ satisfies homogeneous boundary conditions.

   $i)$ $\dfrac{\partial u}{\partial x}(0,t) = A(t),\ u(L,t) = B(t)$ $\qquad (a)$ $u_0(x,t) = xA(t) + \frac{x^2}{2L}[B(t) - A(t)]$

   $ii)$ $u(0,t) = A(t),\ u(L,t) = B(t)$ $\qquad (b)$ $u_0(x,t) = B(t) + A(t)[x - L]$

   $iii)$ $\dfrac{\partial u}{\partial x}(0,t) = A(t),\ \dfrac{\partial u}{\partial x}(L,t) = B(t)$ $\qquad (c)$ $u_0(x,t) = A(t) + \frac{x}{L}[B(t) - A(t)]$

2. Let $u(x,t)$ be the solution of the initial/boundary value problem for the heat equation: $u_t = u_{xx}$ for $0 < x < 1$, $t > 0$, $u(x,0) = 0$, $u(0,t) = 0$, $u(1,t) = 1$.

   a) Compute $v(x)$, where $v(x) = \lim_{t\to\infty} u(x,t)$ (assuming it exist).

   b) Find $u(x,t)$. [HINT: First find $w(x,t) = u(x,t) - v(x)$].

   c) Use the equilibrium solution and one term from the Fourier series of $u(x,t)$ to estimate how long, $t = T$, from $t = 0$ it will take for $u(\frac{1}{2},t)$ to attain half its equilibrium value (in other words, estimate the value of T such that

   $$u(\tfrac{1}{2},t) < \tfrac{1}{2}\lim_{t\to\infty} u(\tfrac{1}{2},t) \quad \text{for } t < T$$

   and

   $$u(\tfrac{1}{2},t) > \tfrac{1}{2}\lim_{t\to\infty} u(\tfrac{1}{2},t) \quad \text{for } t > T.$$

3. Find the solution $u(x,t)$ of the PDE

   $$u_{tt} + 2u_t = u_{xx}$$

   for x in the interval $-1 \le x \le 1$ subject to periodic boundary conditions and also the initial conditions

   $$u(x,0) = 0, \qquad u_t(x,0) = \cos \pi x + 3 \sin 3\pi x.$$

   Your coefficients should be fully simplified.

4. Say $u(x,t)$ is a solution of the heat equation $u_t = u_{xx}$ for $0 \le x \le L$ with boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. Assume the initial temperature is $u(x,0) = f(x)$ is symmetric about the mid-point, that is,

$$f(\tfrac{L}{2} - x) = f(\tfrac{L}{2} + x), \qquad 0 \le x \le L/2.$$

Show that the solution is also symmetric about the mid-point.

5. Let L be the differential operator defined by the formula

$$Lu = u_t - u_{xx}$$

a) Show that the function $u_0 = \frac{1}{2}x(x-1)$ is a particular solution of $Lu_0 = -1$

b) Use this – and linearity – to describe how you would solve $Lu = -1$ on $0 < x < 1$ for all $t > 0$ subject to the initial condition $u(x,0) = 0$ with the homogeneous Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$.

c) Use the procedure you just described to actually find the solution (but don't take time to compute the Fourier coefficients).

6. In the unit disk, $r^2 \le 1$ in the plane, consider the *inhomogeneous* Laplace equation

$$\nabla^2 u = r^2 \quad \text{with} \quad u(1,\theta) = \cos 2\theta.$$

a) Find a particular solution $v(r)$ of this inhomogeneous equation with $v(1) = 0$. [SUGGESTION: As written just above, seek a $v(r)$ that does not depend on θ.]

b) What procedure should you use to find a solution $w(r,\theta)$ of the *homogeneous equation*

$$\nabla^2 w = 0$$

with appropriate boundary conditions so that $u = u_0 + w$ solves the original problem?

c) Carry out the above procedure to solve the original problem for $u(r,\theta)$.

7. Let $f(x) = (1-x)^2$, $0 < x \le 1$.

a) Draw a sketch of the even extension of f to $-1 \le x \le 1$. Then find a "formula" for the extended function.

b) Repeat the above, only this time extend f as an odd function.

8. a) Write e^x as the sum of an even and an odd function of x.

b) More generally, write any function $f(x)$ as the sum of an even function, $\varphi(x)$ and and odd function, $\psi(x)$, so $f(x) = \varphi(x) + \psi(x)$. Your goal is to find simple formulas for φ and ψ in terms of $f(x)$ and $f(-x)$.

9. [In the following, all of the functions should be extended so that they are 2π periodic.]

Say you have computed the Fourier series for $f(x) = \begin{cases} -1 & \text{for } -\pi < x < 0 \\ 1 & \text{for } \quad 0 < x < \pi \end{cases}$.

Let $g(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } \quad 0 < x < \pi \end{cases}$. How are the Fourier series for f and g related?

Same question for $h(x) = \begin{cases} 0 & \text{for } -\pi < x < -\pi/2 \\ 1 & \text{for } -\pi/2 < x < \pi/2 \\ 0 & \text{for } \pi/2 < x < \pi \end{cases}$

10. One useful approach to solving equations like

$$u_t + a(t)u = u_{xx}$$

is to try to find a change of variable, $u(x,t) = \varphi(t)v(x,t)$ to reduce it to the standard heat equation, $v_t = v_{xx}$. Here one want to find $\varphi(t)$ so that the equation for v is simpler.

a) Carry this out for $u_t + 2tu = u_{xx}$.

b) Carry this out for the general case $u_t + a(t)u = u_{xx}$.

11. Solve the wave equation $u)tt = c^2 u_{xx}$ for a string on $-\pi \le x \le \pi$ with fixed ends $u(\pm\pi, t) = 0$ assuming that initially it is plucked at its mid-point:

$$u(x,0) = 1 - \frac{|x|}{\pi}, \qquad \text{and} \qquad u_t(x,0) = 0.$$

[Remark: If you prefer, you may regard the string as on the interval $0 \le z \le L$ with the end points at $z = 0$ and $z = L$ fixed, and plucked at its mid-point $z = L/2$ with initial velocity 0. Except for the choice of coordinates, this is of course the same problem.]

[Last revised: February 25, 2015]