Math 241, Spring 2015 **Jerry L. Kazdan**

Problem Set 4

DUE: Thurs. Feb. 12 in class. [Late papers will be accepted until 1:00 PM Friday.]

This week. Please read all of Chapter 4 Sec. 4 and Chapter 3 Sec. 3.1 - 3.3 in the Haberman text.

Note: Exam 1 Tues, Feb. 17, 10:30-11:50. Closed book, no calculators, no cell phones, but you may use one 3×5 card with notes on both sides.

REMARK: One goal of the Bonus Problem 1 below is to show that if $\phi(t)$ satisfies

$$\phi'' + \gamma\phi = 0,$$

. where γ is a const and and if ϕ satisfies the *periodic boundary conditions*

$$\phi(P) = \phi(0) \quad \text{and} \quad \phi'(P) = \phi'(0), \tag{1}$$

then $\phi(t)$ is periodic with period P, that is, $\phi(t+P) = \phi(t)$ for all t. This justifies why (1) are called "periodic boundary conditions."

1. p. 83 #2.5.10

2. p. 84 #2.5.13

3. p. 84 #2.5.14

4. Solve the Laplace equation in the annular region $1 < r < 2$ in the plane with boundary conditions (polar coordinates) $u(1,\theta) = 3$ and $u(2,\theta) = 5$.

5. [SOLID MEAN VALUE PROPERTY] Let u satisfy the Laplace equation $\Delta u = 0$ in a region Ω in the plane $\mathbb{R}^2$. The book (p. 79) ahows that the for any disk D, the value at the origin is the average of its values on the bounding circle C of that disk.

   Multiplying this formula by r and then integrating, obtain the "Solid Mean Value Property" for a disk $D(a)$ of radius a

   $$u(0,\theta) = \frac{1}{\pi a^2} \int_0^a \left(\int_0^{2\pi} u(r,\theta)\, d\theta \right) r\, dr = \frac{1}{\pi a^2} \iint_{D(a)} u(r,\theta)\, dA,$$

   where $dA = r\, dr\, d\theta$ is the element of area in polar coordinates.

6. p. 143 # 4.4.6

7. Use separation of variables to solve the wave equation $u_{tt} = u_{xx}$ (so we are taking $c = 1$ for a vibrating string $0 < x < \pi$ with initial conditions

$$u(x,0) = 3\sin 7x + 2\sin 19x, \qquad u_t(x,0) = 8\sin 5x,$$

and boundary conditions (fixed end points) $u(0,t) = 0,\ \ u(\pi,t)=0$.

8. a) Let $u(x,t)$ be a solution of the wave equation $u_{tt} = u_{xx}$ for a vibrating string $0 < x < L$ whose end points are fixed: $u(0,t) = 0,\ \ u(L,t) = 0$ and define the "Energy", $E(t)$, by

$$E(t) = \frac{1}{2}\int_0^L (u_t^2 + u_x^2)\,dx.$$

Show that energy is conserved, $dE/dt = 0$. [Suggestion: Integrate by parts. Since $u = 0$ on the boundary, then also $u_t = 0$ there.]

b) If $u(x,0) = 0$ and $u_t(x,0) = 0$, what can you conclude?

c) Generalize this to the motion $u(x,y,t)$ of a vibrating membrane, $\Omega \subset \mathbb{R}^2$ so $u_{tt} = \Delta u$ in Ω whose boundary, $\partial\Omega$. is fixed: $u(x,y,t) = 0$ for all (x,y) on $\partial\Omega$ and all $t \geq 0$. Here

$$E(t) = \frac{1}{2}\iint_D (u_t^2 + |\nabla u|^2)\,dx\,dy.$$

Show that energy is conserved.

Bonus Problem

[Please give this directly to Professor Kazdan]

B-1 Say a function $u(t)$ satisfies the differential equation

$$u'' + b(t)u' + c(t)u = 0 \qquad (2)$$

on the interval $[0,\,A]$ and that the coefficients $b(t)$ and $c(t)$ are both bounded, say $|b(t)| \leq M$ and $|c(t)| \leq M$ (if the coefficients are continuous, this is always true for some M).

a) Define $E(t) := \frac{1}{2}(u'^2 + u^2)$. Show that for some constant γ (depending on M) we have $E'(t) \leq \gamma E(t)$. [Suggestion: use the simple inequality $2xy \leq x^2 + y^2$.]

b) Show that $E(t) \leq e^{\gamma t}E(0)$ for all $t \in [0,\,A]$. [Hint: First use the previous part to show that $(e^{-\gamma t}E(t))' \leq 0$].

c) In particular, if $u(0) = 0$ and $u'(0) = 0$, show that $E(t) = 0$ and hence $u(t) = 0$ for all $t \in [0,\,A]$. In other words, if $u'' + b(t)u' + c(t)u = 0$ on the interval $[0,\,A]$ and that the functions $b(t)$ and $c(t)$ are both bounded, and if $u(0) = 0$ and $u'(0) = 0$, then the only possibility is that $u(t) \equiv 0$ for all $t \geq 0$.

d) Use this to prove the *uniqueness theorem*: if $v(t)$ and $w(t)$ both satisfy equation

$$u'' + b(t)u' + c(t)u = f(t) \tag{3}$$

and have the same initial conditions, $v(0) = w(0)$ and $v'(0) = w'(0)$, then $v(t) \equiv w(t)$ in the interval $[0, A]$.

e) Assume the coefficients $b(t)$, $c(t)$, and $f(t)$ in equation (3) are periodic with period P, that is, $b(t + P) = b(t)$ etc. for all real t. If $\phi(t)$ is a solution of equation (3) that satisfies the *periodic boundary conditions* (1), show that $\phi(t)$ is periodic with period P.

[Last revised: June 28, 2015]