Problem Set 3 #2.5.9b

NOTE: In working Homework Set 3 #11 = page 83: 2.5.9b in office hours, it became clear that it requires facts covered in Math 114 and 240 that few people seemed to remember. Here is a brief summary.

Review 1. Say you have a square matrix and want to solve the homogeneous equation $AX = 0$. Clearly a "trivial" solution is $X = 0$. FACT: There is a non-trivial solution $X \neq 0$ if and only if $\det A = 0$. This is particularly simple to use if A is a 2×2 matrix. For larger matrices the determinant is often unpleasant – or essentially impossible – to compute.

Review 2. If $z = \alpha + i\beta$ is a complex number and $r > 0$, how do you compute r^z? Since

$$e^{\alpha+i\beta} = e^{\alpha}e^{i\beta} = e^{\alpha}(\cos\beta + i\sin\beta), \tag{1}$$

we can compute r^z for the special case when $r = e$. For the general case we use the *definition*

$$\begin{aligned} r^z =& e^{z\ln r} = e^{\alpha\ln r}e^{i\beta\ln r} \\ =& r^{\alpha}[\cos(\beta\ln r) + i\sin(\beta\ln r)] \end{aligned} \tag{2}$$

Review 3: Euler's Differential Equation The homogenyous equation is

$$Ly =: ax^2y'' + bxy' + cy = 0. \tag{3}$$

To solve this we try $y = x^q$, where q might be a complex number (see REVIEW 2 above. Then by a straight forward computation,

$$\begin{aligned} L(x^q) &= ax^2q(q-1)x^{q-2} + bxqx^{q-1} + cx^q \\ &= [aq(q-1) + bq + c]x^{\alpha} \end{aligned}$$

Thus, if q is a root of the quadratic polynomial $P(q) = aq(q-1) + bq + c$, then $L(x^q) = 0$ so $y = x^q$ is a solution of the homogeneous equation $Ly = 0$. Usually, the quadratic polynomial $P(q) = 0$ has two *distinct* roots, say q_1 and q_2 (which might be complex!). In that case the general solution of the homogeneous equation $Ly = 0$ is

$$y = Ax^{q_1} + Bx^{q_2}$$

where A and B are constants. If the coefficients a, b. and c are real, then complex roots of $P(q) = 0$ occur in conjugate pairs, $q = \alpha \pm i\beta$. Then by equation (2) the general real form of the solution of $Ly = 0$ is

$$y = Ar^{\alpha}\cos(\beta\ln x) + Br^{\alpha}\sin(\beta\ln x) \tag{4}$$

If q happens to be a double (that is, repeated) root of $P(q) = 0$, then the general solution of $Ly = 0$ is

$$y = Ax^q + Bx^q \ln x.$$

Problem 2.5.9b Solve the Laplace equation in the circular sector (polar coordinates!) $0 < a < r < b,\ 0 < \theta < \pi/2$ with the boundary conditions:

$$u(r, 0) = 0, \quad u(r, \pi/2) = f(r), \qquad u(a, \theta) = 0, \quad u(b, \theta) = 0$$

Note that in polar coordinates the Laplace equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Using separation of variables we seek a solution of the form $u(r, \theta) = \phi(\theta)G(r)$. A routine computation then gives

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \text{const } = \gamma.$$

The boundary conditions give

$$0 = u(r, 0) = \phi(0)G(r), \qquad f(r) = u(r, \pi/2) = \phi(\pi/2)G(r)$$

$$0 = u(a, \theta) = \phi(\theta)G(a), \quad 0 = u(b, \theta) = \phi(\theta)G(b).$$

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