Problem Set 2

Due: Thurs. Jan. 29 in class. [Late papers will be accepted until 1:00 PM Friday.]

This week. Please read all of Chapter 2 in the Haberman text.

1. Say the temperature in the plane $\mathbb{R}^2$ at time t is given by

$$u(x,y,t) = 1 - x + 3y^2 + xyt.$$

   If you are at the point $(1,2)$ at time $t=5$, in what direction in $\mathbb{R}^2$ should you move so the the temperature increases most? Decreases most?

2. Let $\mathbf{V} = (x+2y^2)\mathbf{i} + (2+3y\mathbf{j}$ be a vector field in the plane and let D be the unit disk whose boundary, B is of course the unit circle. Use the divergence theorem to compute

$$\oint_B \mathbf{V}\cdot\mathbf{N}\,ds,$$

   where $\mathbf{N}$ is the unit outer normal vector field on B.

3. Let $\mathbf{x}$ be a point in $\mathbb{R}^3$ and the vector field $\mathbf{F}(\mathbf{x})$ have continuous first derivatives. Assume $\mathbf{F}$ decays for large $|\mathbf{x}|$:

$$|\mathbf{F}(\mathbf{x})| \le \frac{1}{1+|\mathbf{x}|^3}$$

   for all $\mathbf{x}$. Show that

$$\iiint_{\mathbb{R}^3} \nabla\cdot\mathbf{F}\,dx\,dy\,dz = 0$$

   Hint: First work with a large ball, $|\mathbf{x}| \le r$, apply the divergence theorem, and then let $r\to\infty$. Note that on the sphere of radius r the unit outer normal is just a radial vector, $|\mathbf{F}\cdot\mathbf{N}| \le |\mathbf{F}|$, and $d\text{Area} = r^2\,d\omega$, where $d\omega$ is the element of area on the *unit* sphere [on the *ball* of radius r, the element of volume in spherical coordinates is $dVol = r^2 dr\,d\omega$.]

4. p. 18, #1.4.7 in Haberman.

   Remark: For an equilibrium temperature distribution to exist, $u_t(x,t)$ must tend to 0 as $t\to\infty$. If the Total Thermal Energy, $H(t) = \int_0^L u(x,t)\,dx$ (see #5 below) this implies that $dH/dt \to 0$ as $t\to\infty$. From the data specified you can compute dH/dt as well as the initial value, $H(0)$.

5. p. 19, #1.4.11 in Haberman.

   REMARKS: In the text, the standard version of the heat equation in one space dimension is given by equation (1.2.9) on page 8:

   $$c\rho u_t = (K u_x)_x + Q(x,t,u),$$

   where Q describes the internal sources of heat energy. In this problem , the heat equation is:

   $$u_t = u_{xx} + x$$

   Comparing them we see that $c\rho = 1$, $K = 1$, and $Q = x$.

   Thus

   $$H(t) = \text{ Total Thermal Energy } = \int_0^L c\rho u(x,t)dx = \int_0^L u(x,t)dx.$$

   From the given data one can compute dH/dt and also $H(0)$. Using this you can compute $H(t)$.

6. p. 28, #1.5.8 in Haberman.

7. p. 28, #1.5.9 in Haberman.

   [Polar coordinates!. Seek the solution as a function only of the radius r.]

8. p. 28, #1.5.11 in Haberman.

   REMARK: Note that the right side of this equation is just the Laplacian in polar coordinates in the plane for a function $u(r)$ that does not depend on the angle θ [compare eq (1.5.19) in Haberman]. You will need that in polar coordinates the element of area $dA = r\,dr\,d\theta$. The total thermal energy $H(t) = \iint u(r,t)\,dA$ integrated over the annulus. Begin by computing dH/dt using the given data.

9. p. 29, #1.5.13 in Haberman.

   [Use spherical coordinates, Eq. (1.5.22). Seek a solution $u(r)$ that depends only on the radius.]

10. p. 34, #2.2.4 in Haberman.

11. p. 51, #2.3.2(a-f) in Haberman.

    SUGGESTION: For (f), first do the special case $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ in the interval $0 < x < L$. Then for $\frac{d^2\phi}{dy^2} + \lambda\phi = 0$ in the more general interval $a < y < b$ make the change of variable $x = y - a$ and let $L = b - a$ be the length of the interval. This reduces the problem to the special case.

[Last revised: January 29, 2015]