Math 241, Spring 2015 **Jerry L. Kazdan**

Problem Set 11

DUE: Thurs. April 23 in class. [Late papers will be accepted until 1:00 PM Friday.]

This week: In the Haberman text, please read Chapter 10 (lightly).

1. This problem (see Haberman p. 362-364) is an example of solving

$$u_{tt} = c^2 \Delta u + h(\vec{x}) \cos \omega t$$

where $\vec{x}$ is a point in a bounded set $\Omega \subset \mathbb{R}^n$ and $u = 0$ on $\partial\Omega$.

Let Ω be the real interval $0 \le x \le \pi$. *Explicitly* solve

$$u_{tt} = u_{xx} + (\sin 2x + 5 \sin 3x) \cos 2t$$

with

$$u(0,t) = u(\pi,t) = 0, \qquad \text{and} \quad u(x,0) = 0, \quad u_t(x,0) = 0.$$

What can you say about the solution $u(x,t)$ as $t \to \infty$?

NOTE: For the first part you will need to find particular solutions $y_p(t)$ of the inhomogenuous equation $y''(t) + n^2 y = \gamma \cos kt$. There are two cases:

Case 1: $n \ne k$: Then $y_p(t) = \gamma \dfrac{\cos kt}{n^2 - k^2}$

Case 2: $n = k$: Then $y_p(t) = \gamma \dfrac{t \sin kt}{2k}$ (resonance)

This repeats information given in the text, page 362-363, equations (8.5.26) – (8.5.31).

2. [DELTED Too many people found this too complicated.]

Consider the following boundary value problem on the interval $\{-1 \le x \le 1\}$

$$u''(x) = f(x) \quad \text{with} \quad u(-1) = u(1) = 0.$$

By direct integration (with your bare hands), find a formula of the form

$$u(x) = \int_{-1}^{1} G(x,y) f(y)\, dy,$$

Thus, the problem is to find $G(x,y)$. [We did this in class for the interval $0 \le x \le 1$].

ANSWER: $G(x,y) = \frac{1}{2}\big[|x-y| + xy - 1\big]$

3. In a bounded domain $\Omega \subset \mathbb{R}^n$ consider the boundary value problem of solving $\Delta u = f(\vec{x})$ with the homogeneous Neumann boundary condition $\nabla u \cdot \mathbf{N} = 0$ on the boundary, $\partial\Omega$.

a) Show that if a solution exists, then f must satisfy

$$\iint_\Omega f(\vec{x})\, \mathrm{dVol} \ = 0. \tag{1}$$

Here, dVol is the element of "Volume" in $\mathbb{R}^n$.

b) In the special case $n = 1$ where Ω is the interval $\{0 \le x \le 1\}$, use direct integration of $u'' = f(x)$ to show that if f satisfies (1), then this boundary value problem has a solution by finding some function $N(x, y)$ so that

$$u(x) = \int_0^1 N(x, y) f(y)\, dy.$$

To what extent is the solution of this problem unique?

4. Let $\Omega \subset \mathbb{R}^2$ be a region inside the rectangle with vertices at $(-1, -1)$, $(2, -1)$, $(2, 2)$, and $(-1, 2)$, and assume the square with vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$ is inside Ω. Use this information to estimate the lowest eigenvalue of the Laplacian for the region Ω with boundary values zero. Thus, find numbers $0 < m < M$ so that

$$m < \lambda_1(\Omega) < M.$$

Bonus Problem

[Please give this directly to Professor Kazdan]

B-1 Let Ω in $\mathbb{R}^2$ be a bounded region and let $\hat{\Omega} \subset \mathbb{R}^2$ be the region obtained by stretching the x and y coordinates by a factor $c > 0$. Thus $\hat{x} = cx$ and $\hat{y} = cy$. The λ_n and v_n be the eigenvalues and corresponding eigenfunctions of Ω.

a) What can you say about the eigenvalues and eigenfunctions of $\hat{\Omega}$?

b) Repeat this for a region Ω in $\mathbb{R}^3$.

[Last revised: April 20, 2015]