Problem Set 1

DUE: Thurs. Jan. 22 in class. [Late papers will be accepted until 1:00 PM Friday.]

This is rust remover. It is essentially Homework Set 0 with a few modifications. NOTATION: $u_t = \dfrac{\partial u}{\partial t}$.

This week. Please read all of Chapter 1 in the Haberman text.

1. Let $u(t)$ be the solution of $u' = 3u$ with initial value $u(0) = A > 0$. At what time T is $u(T) = 2A$?

2. Let $u(t)$ be the amount of a radioactive element at time t and say initially, $u(0) = A > 0$. The rate of decay is proportional to the amount present, so

$$\frac{du}{dt} = -cu,$$

where the constant $c > 0$ determines the decay rate. The *half-life* T is the amount of time for half of the element to decay, so $u(T) = \frac{1}{2}u(0)$. Find c in terms of T and obtain a formula for $u(t)$ in terms of T.

3. Let $\displaystyle\int_0^x f(t)\,dt = e^{\cos(3x)} + A$, where f is some continuous function. Find f and the constant A.

4. a) If $u'' + 4u = 0$ with initial conditions $u(0) = 1$ and $u'(0) = -2$, compute $u(t)$.
   b) Find a particular solution of the inhomogeneous equation $u'' + 4u = 8$.
   c) Find a particular solution of the inhomogeneous equation $u'' + 4u = -4t$.
   d) Find a particular solution of the inhomogeneous equation $u'' + 4u = -8 - 8t$.
   e) Find the most general solution of the inhomogeneous equation $u'' + 4u = 8 - 8t$.
   f) If $f(t)$ is any continuous function, use the method "variation of parameters" (look it up if you don't know it) to find a formula for a particular solution of $u'' + 4u = f(t)$.

5. Let $u(t)$ be any solution of $u'' + 2bu' + 4u = 0$. If $b > 0$ is a constant, show that $\lim_{t\to\infty} u(t) = 0$.

6. a) If $u'' - 4u = 0$ with initial conditions $u(0) = 1$ and $u'(0) = -2$, compute $u(t)$.
   b) Find a particular solution of the inhomogeneous equation $u'' - 4u = 8$.
   c) Find a particular solution of the inhomogeneous equation $u'' - 4u = -4t$.

d) Find a particular solution of the inhomogeneous equation $u'' - 4u = -8 - 8t$.

e) Find the most general solution of the inhomogeneous equation $u'' - 4u = 8 - 8t$.

f) If $f(t)$ is any continuous function, use the method "variation of parameters" (look it up if you don't know it) to find a formula for a particular solution of $u'' - 4u = f(t)$.

7. Say $w(t)$ satisfies the differential equation

$$aw''(t) + bw' + cw(t) = 0, \tag{1}$$

where a and c, are positive constants and $b \geq 0$. Let $E(t) = \frac{1}{2}[aw'^2 + cw^2]$.

a) Without solving the differential equation, show that $E'(t) \leq 0$.

b) Use this to show that If you also know that $w(0) = 0$ and $w'(0) = 0$, then $w(t) = 0$ for all $t \geq 0$.

c) [Uniqueness] Say the functions $u(t)$ and $v(t)$ both satisfy the same equation (1) and also $u(0) = v(0)$ and $u'(0) = v'(0)$. Show that $u(t) = v(t)$ for all $t \geq 0$.

8. Say $u(x,t)$ has the property that $\dfrac{\partial u}{\partial t} = 2$ for all points $(x,t) \in \mathbb{R}^2$.

a) Find some function $u(x,t)$ with this property.

b) Find the most general such function $u(x,t)$.

c) If $u(x,0) = \sin 3x$, find $u(x,t)$.

d) If instead u satisfies $\dfrac{\partial u}{\partial t} = 2xt$, still with $u(x,0) = \sin 3x$, find $u(x,t)$.

9. Say $u(x,t)$ has the property that $\dfrac{\partial u}{\partial t} = 3u$ for all points $(x,t) \in \mathbb{R}^2$.

a) Find some such function – other than the trivial $u(x,t) \equiv 0$.

b) Find the most general such function.

c) If $u(x,t)$ also satisfies the initial condition $u(x,0) = \sin 3x$, find $u(x,t)$.

10. a) If $u(x,t) = \cos(x - 3t) + 2(x - 3t)^7$, show that $3u_x + u_t = 0$.

b) If $f(s)$ is *any* smooth function of s and $u(x,t) = f(x - 3t)$, show that $3u_x + u_t = 0$.

11. A function $u(x,y)$ satisfies $3u_x + u_t = f(x,t)$, where f is some specified function.

a) Find an invertible linear change of variables

$$\begin{aligned} r &= ax + bt \\ s &= cx + dt, \end{aligned}$$

where a, b, c, d are constants, so that in the new $(r,\ s)$ variables u satisfies $\dfrac{\partial u}{\partial s} = g(r,s)$, where g is related to f by the change of variables. [REMARK: There are many possible such changes of variable. The point is to reduce the differential operator $3u_x + u_t$ to the much simpler u_s.]

b) Use this procedure to solve

$$3u_x + u_t = 1 + x + 2t \qquad \text{with} \qquad u(x,0) = e^x.$$

12. Let S and T be linear spaces, such as $\mathbb{R}^3$ and $\mathbb{R}^7$ and $L : S \to T$ be a *linear map*; thus, for any vectors X, Y in S and any scalar c

$$L(X+Y) = LX + LY \quad \text{and} \quad L(cX) = cL(x).$$

Say V_1 and V_2 are (distinct!) solutions of the equation $LX = Y_1$ while W is a solution of $LX = Y_2$. Answer the following in terms of V_1, V_2, and W.

a) Find some solution of $LX = 2Y_1 - 7Y_2$.

b) Find another solution (other than W) of $LX = Y_2$.

13. The following is a table of inner ("dot") products of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

	$\mathbf{u}$	$\mathbf{v}$	$\mathbf{w}$
$\mathbf{u}$	4	0	8
$\mathbf{v}$	0	1	3
$\mathbf{w}$	8	3	50

For example, $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} = 3$.

a) Find a unit vector in the same direction as $\mathbf{u}$.

b) Compute $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$.

c) Compute $\|\mathbf{v} + \mathbf{w}\|$.

d) Find the orthogonal projection of $\mathbf{w}$ into the plane E spanned by $\mathbf{u}$ and $\mathbf{v}$. [Express your solution as linear combinations of $\mathbf{u}$ and $\mathbf{v}$.]

e) Find a unit vector orthogonal to the plane E.

f) Find an orthonormal basis of the three dimensional space spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

14. Let z and w be complex numbers.

a) Write the complex number $z = \dfrac{1}{3+4i}$ in the form $z = a + ib$ where a and b are real numbers.

b) Show that $\overline{(zw)} = \bar{z}\bar{w}$.

c) Show that $|z|^2 = z\bar{z}$.

d) show that $|zw| = |z||w|$.

15. If $z = x + iy$ is a complex number, one way to define e^z is by the power series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^k}{k!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{2}$$

a) Using the usual (real) power series for $\cos y$ and $\sin y$, show that

$$e^{iy} = \cos y + i \sin y.$$

b) Use this to show that $\cos y = \dfrac{e^{iy} + e^{-iy}}{2}$ and $\sin y = \dfrac{e^{iy} - e^{-iy}}{2i}$.

c) Using equation (2), one can show that $e^{z+w} = e^z e^w$ for any complex numbers z and w (accept this for now). Consequently

$$e^{i(x+y)} = e^{ix} e^{iy}.$$

Use the result of part (a) to show that this implies the usual formulas for $\cos(x+y)$ and $\sin(x+y)$.

16. Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded (connected) region with smooth boundary $\mathcal{B}$. If $u(x, y)$ is a "smooth" function, write $\Delta u = u_{xx} + u_{yy}$ (we call Δ the *Laplace operator*). Some people write $\Delta u = \nabla^2 u$.

SUGGESTION: First do this problem for a function of *one* variable, $u(x)$, so $\Delta u = u''$ and, say, $\mathcal{D}$ is the interval $\{0 < x < 1\}$.

a) Show that $u\Delta u = \nabla \cdot (u\nabla u) - |\nabla u|^2$.

b) If $u(x, y) = 0$ on $\mathcal{B}$. Show that

$$\iint_{\mathcal{D}} u\Delta u \, dx\, dy = -\iint_{\mathcal{D}} |\nabla u|^2 \, dx\, dy.$$

c) If $\Delta u = 0$ in $\mathcal{D}$ and $u = 0$ on the boundary $\mathcal{B}$, show that $u(x, y) = 0$ throughout $\mathcal{D}$.

17. The temperature $u(x, t)$ of a certain thin rod, $0 \le x \le L$ satisfies the *heat equation*

$$u_t = u_{xx}$$

Assume the initial temperature $u(x,0) = 0$ and that both ends of the rod are kept at a temperature of 0, so $u(0,t) = u(L,t) = 0$ for all $t \ge 0$. What do you anticipate the temperature in the rod will be at any later time t?

I hope you suspect that $u(x,t) = 0$ for all $t \ge 0$. Use the following to prove this. Let

$$H(t) = \int_0^L u^2(x,t)\,dx.$$

a) Show that since the temperature on the ends of the rod is always zero, then $dH/dt \le 0$ (an integration by parts will be needed). Thus, for any $t \ge 0$ we know that $H(t) \le H(0)$

b) Since the initial temperature is zero, what is $H(0)$? Why does this imply that $H(t) = 0$ for all $t \ge 0$? Why does this imply that $u(x,t) = 0$ for all points on the rod and all $t \ge 0$?

Bonus Problem

[Please give this directly to Professor Kazdan]

B-1 [Generalization of Problem 17 to more space dimensions]. Say a function $u(x,y,t)$ satisfies the heat equation in a bounded region $\Omega \in \mathbb{R}^2$:

$$u_t = u_{xx} + u_{yy}$$

and that $u(x,y,t) = 0$ for all points (x,y) on the boundary, $\mathcal{B}$ of Ω and all $t \ge 0$. Similar to Problem 17, define

$$H(t) = \iint_\Omega u^2(x,y,t)\,dx\,dy.$$

a) Use $u(x,y,t) = 0$ for all points (x,y) on the boundary $\mathcal{B}$ and all $t \ge 0$ to show that $dH/dt \le 0$ for all $t \ge 0$. [SUGGESTION: See Problem 16.]

b) If in addition use that the initial temperature is zero, $u(x,y,0) = 0$ for all points $(x,y) \in \Omega$, to show that $u(x,y,t) = 0$ for all $(x,y) \in \Omega$ and all $t \ge 0$.

[Last revised: January 23, 2015]