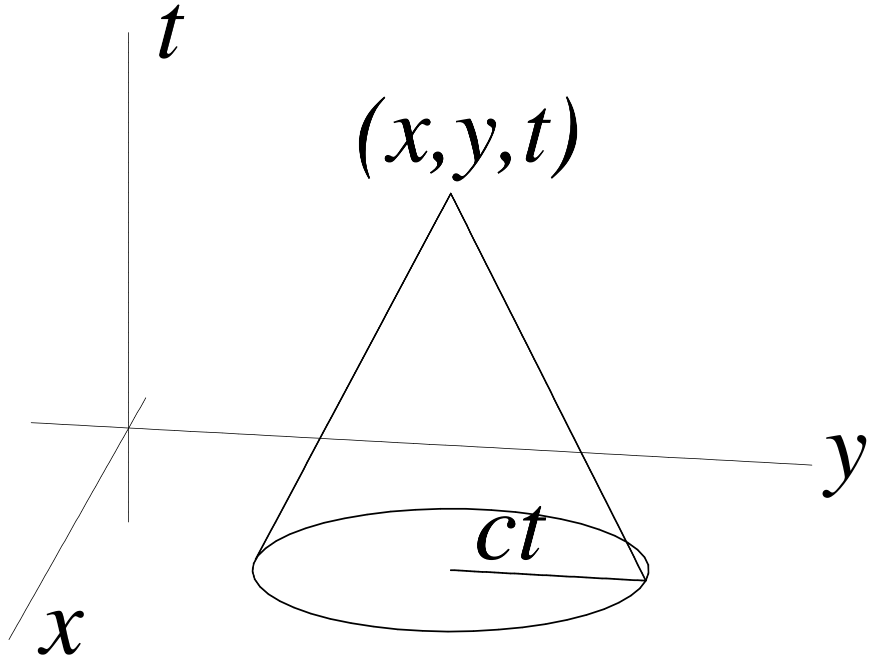
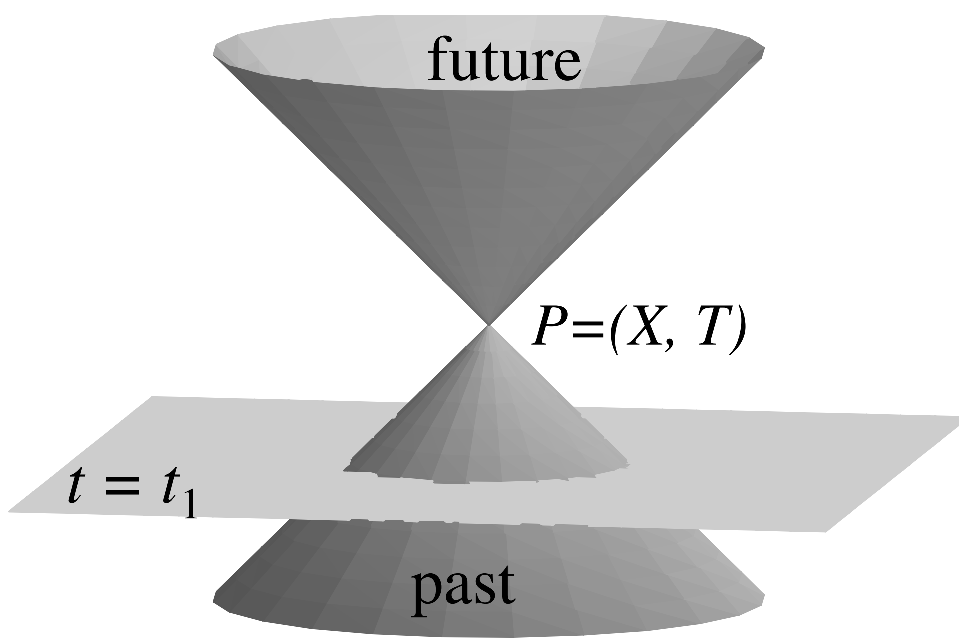
Light Cones





Formulas for the solution of the Homogeneous Wave Equation

$$u_{tt} = \Delta u, \qquad \text{with} \quad u(\mathbf{x},0) = f(\mathbf{x}) \quad \text{and} \quad u_t(\mathbf{x},0) = g(\mathbf{x}). \tag{1}$$

Let $\mathbf{x} = (x, y)$. For the two (space) dimensional wave equation it is

$$u(x,y,t) = \frac{1}{2\pi c}\frac{\partial}{\partial t}\left(\iint\limits_{r\le ct} \frac{f(\xi,\eta)}{\sqrt{c^2t^2 - r^2}}\, d\xi\, d\eta\right) + \frac{1}{2\pi c}\iint\limits_{r\le ct} \frac{g(\xi,\eta)}{\sqrt{c^2t^2 - r^2}}\, d\xi\, d\eta, \tag{2}$$

where $r^2 = (x-\xi)^2 + (y-\eta)^2$.

In three (space) dimensions, $\mathbf{x} = (x, y, z)$, one has

$$u(x,y,z,t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\iint\limits_{r=ct} f(\xi,\eta,\zeta)\, dA\right) + \frac{1}{4\pi c^2 t}\iint\limits_{r=ct} g(\xi,\eta,\zeta)\, dA, \tag{3}$$

where $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$ and dA is the element of surface area on the sphere centered at (x, y, z) with radius $r = ct$.

These are called *Kirchoff's formulas.* It is simplest first to obtain the formula in the three space dimensional case (3), and then obtain the two dimensional case (2) from the special three dimensional case where the initial data $f(x, y, z)$ and $g(x, y, z)$ are independent of z. This observation is called *Hadamard's method of descent.*

Inhomogeneous Equation

There are also formulas for the solution of the inhomogeneous wave equation

$$Lu := u_{tt} - c^2\Delta u = F(x,t). \tag{4}$$

The approach is analogous to Lagrange's method of variation of parameters, which gives a formula for the solution of an inhomogeneous equation such as $u'' + u = F(t)$ in terms of solutions of the homogeneous equation. The method is called *Duhamel's principle.*
We illustrate it for the wave equation, seeking a solution of (4) with initial conditions

$$u(x,0) = 0 \quad u_t(x,0) = 0.$$

Since we are solving a differential equation, it is plausible to find a solution as an integral in the form

$$u(x,t) = \int_0^t v(x,t;s)\, ds \tag{5}$$

where the function $v(x, t; s)$, which depends on a parameter s, is to be found. This clearly already satisfies the initial condition $u(x, 0) = 0$. Working formally, we have

$$u_t(x,t) = \int_0^t v_t(x,t;s)\, ds + v(x,t;t),$$

so $u_t(x,0) = 0$ implies $v(x,0;0) = 0$. In fact, we will further restrict v by requiring that $v(x,t;t) = 0$ for all $t \geq 0$. Then the formula for u_t simplifies and

$$u_{tt}(x,t) = \int_0^t v_{tt}(x,t;s)\,ds + v_t(x,t;s)\big|_{s=t}.$$

The similar formula for Δu is obvious. Substituting these into the wave equation (4) we want

$$F(x,t) = Lu(x,t) = \int_0^t Lv(x,t;s)\,ds + v_t(x,t;s)\big|_{s=t}.$$

This is evidently satisfied if $Lv = 0$ and $v_t(x,t;s)\big|_{s=t} = F(x,t)$ along with $v(x,t;t) = 0$ for all $t \geq 0$.

Because the coefficients in the wave equation do not depend on t, our results can be simplified a bit by writing $v(x,t;s) = w(x,t-s;s)$ so for each fixed s, the function $w(x,t;s)$ satisfies

$$w_{tt} = c^2 \Delta w \qquad \text{with} \;\; w(x,0,;s) = 0 \;\; \text{and} \;\; w_t(x,0;s) = F(x,s). \tag{6}$$

We can now find w by using our earlier formulas. For instance, in three space variables, from (3)

$$w(x,t;s) = \frac{1}{4\pi c^2 t} \iint\limits_{\|\xi - x\| = ct} F(\xi,s)\,dA_\xi,$$

where dA_ξ is the element of surface area on the sphere centered at x with radius ct, that is, $\|\xi - x\| = ct$. Therefore from (5)

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi c^2} \int_0^t \frac{1}{t-s} \iint\limits_{\|\xi - x\| = c(t-s)} F(\xi,s)\,dA_\xi\,ds \\ &= \frac{1}{4\pi c} \int_0^t \iint\limits_{\|\xi - x\| = c(t-s)} \frac{F(\xi, t - \|\xi - x\|/c)}{\|\xi - x\|}\,dA_\xi\,ds. \end{aligned}$$

But in spherical coordinates, the element of volume $d\xi = c dA_\xi ds$ so we finally obtain

$$u(x,t) = \frac{1}{4\pi c^2} \iiint\limits_{\|\xi - x\| \leq ct} \frac{F(\xi, t - \|\xi - x\|/c)}{\|\xi - x\|}\,d\xi. \tag{7}$$

Thus, to solve the inhomogeneous equation we integrate over backward cone $\|\xi - x\| \leq ct$, which is exactly the domain of dependence of the point (x,t).