The Wave Equation in $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$

Introduction
Light and sound are but two of the phenomena for which the classical wave equation is a reasonable model. This study is one of the real success stories in mathematics and physics. It has led to the development of many valuable techniques.

One space dimension
Upon studying the motion of a vibrating string one is led to the simple differential equation

$$u_{tt} = c^2 u_{xx}, \tag{1}$$

where $u(x,t)$ denotes the displacement of the string at the point x at time t and $c > 0$ is a constant that involves the density and tension of the string. We'll shortly show how to interpret c as the velocity of the propagation of the wave.
By making the change of variables $\xi = x - ct$ and $\eta = x + ct$ in (1), we find

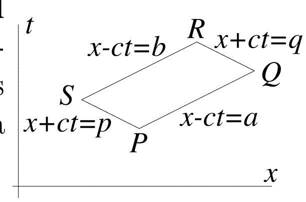
$$u_{\xi\eta} = 0.$$

Integrating this twice reveals the "general" solution $u(\xi, \eta) = f(\xi) + g(\eta)$ for any twice differentiable functions f and g. Untangling the change of variables give us the general solution of (1):

$$u(x,t) = F(x - ct) + G(x + ct). \tag{2}$$

The term $F(x - ct)$ represents a wave traveling to the right with velocity c. We saw this in the previous Section **??** when we discussed the transport equation. The sketches there substantiate the statement that c is the velocity of propagation of the wave. Similarly, $G(x + ct)$ represents a wave traveling to the left with velocity c, so the general solution is composed of waves traveling in both directions. The two families of straight lines $x - ct$ =const, and $x + ct$ =const are the *characteristics* of the wave equation (1).

The formula (2) implies an interesting identity we will need soon. Let P, Q, R, and S be the successive vertices of a parallelogram whose sides consist of the four characteristic lines $x - ct = a$, $x - ct = b$, $x + ct = p$, and $x + ct = q$. If $u(x,t)$ is a solution of the wave equation, then



$$u(P) + u(R) = u(Q) + u(S). \tag{3}$$

This is clear since $u(P) = F(a) + G(p)$, $u(Q) = F(a) + G(q)$, $u(R) = F(b) + G(q)$, and $u(S) = F(b) + G(p)$.

Infinite string, $-\infty < x < \infty$

On physical grounds based on experiments with the motion of particles, we anticipate that we should specify the following *initial conditions*:

$$\begin{array}{ll} \text{initial position} & u(x,0) = f(x) \\ \text{initial velocity} & u_t(x,0) = g(x). \end{array} \tag{4}$$

Using these conditions we can uniquely determine F and G in (2). This gives *d'Alembert's solution* of the initial value problem (1), (4):

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds. \tag{5}$$

Exercise: Consider the equation

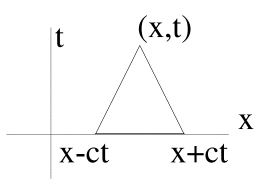
$$u_{xx} - 3u_{xt} - 4u_{tt} = 0. \tag{6}$$

a) Find a change of variable $\xi = ax + bt,\ \eta = cx + dt$ so that in the new coordinates the equation is the standard wave equation

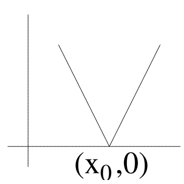
$$u_{\xi\xi} = u_{\eta\eta}.$$

b) Use this to solve (6) with the initial conditions

$$u(x,0) = x^2,\ u_t(x,0) = 2e^x.$$



It is instructive to note that the solution at (x,t) depends only on the initial data in the interval between the points $x - ct$ and $x + ct$. This interval is called the *domain of dependence* of the point (x,t).



Similarly, the initial data at a point $(x_0, 0)$ can only affect the solution $u(x,t)$ for points in the triangular region $|x - x_0| \le ct$. This region is called the *domain of influence* of the point $(x_0, 0)$

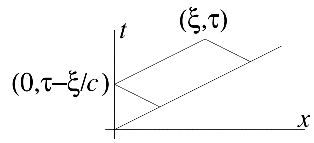
Semi-infinite string, $0 < x < \infty$

Semi-infinite strings can also be treated.

Special Case 1. As an example, we specify zero initial position and velocity but allow motion of the left end point:

$$u(x,0) = 0, \quad u_t(x,0) = 0 \ \text{ for } \ x > 0, \qquad \text{while } \ u(0,t) = h(t) \ \text{ for } \ t > 0. \tag{7}$$

We'll assume that $h(0) = 0$ to insure continuity at the origin.
The critical characteristic $x = ct$ is important here. The domain of dependence of any point to the right of this line does not include the positive t-axis. Thus, if $x \geq ct$, then $u(x,t) = 0$. Next we consider a point (ξ, τ) above this characteristic. The simplest approach is to use the identity (3) with a characteristic parallelogram having its base on the critical characteristic $x = ct$. The characteristic of the form $x - ct =$ const. through (ξ, τ) intersects the t-axis at $t = \tau - \xi/c$.
Since $u(x,t) = 0$ on the base of this parallelogram, then by (3) we conclude that $u(\xi, \tau) = h(\tau - \xi/c)$. To summarize, we see that



$$u(x,t) = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq x \\ h(t - x/c) & \text{for} \quad 0 \leq x \leq t. \end{cases}$$

Special Case 2. A clever observation helps to solve the related problem for a semi-infinite string:

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \ \text{ for } \ x > 0, \qquad \text{while } \ u(0,t) = 0 \ \text{ for } \ t > 0. \quad (8)$$

The observation is that for the infinite string $-\infty < x < \infty$, if the initial position $u(x,0) = f(x)$ and velocity $u_t(x,0) = g(x)$ are odd functions, then so is the solution $u(x,t)$ (proof?). Thus, to solve (8) we simply extend $f(x)$ and $g(x)$ to all of $\mathbb{R}$ as odd functions $f_{odd}(x)$ and $g_{odd}(x)$ and then use the d'Alembert formula (5).

Exercise: Carry this out explicitly for the special case where (8) holds with $g(x) = 0$. In particular, show that for $x > 0$ and $t > 0$

$$u(x,t) = \begin{cases} \frac{1}{2}[f(x + ct) + f(x - ct)] & \text{for} \quad x > ct \\ \frac{1}{2}[f(ct + x) - f(ct - x)] & \text{for} \quad x < ct. \end{cases}$$

The boundary condition at $x = 0$ serves as a reflection. One can see this clearly from a sketch, say with the specific function $f(x) = (x - 2)(3 - x))$ for $2 \leq x \leq 3$ and $f(x) = 0$ for both $0 \leq x \leq 2$ and $x > 3$.

General Case. For a semi-infinite string, the general problem with the initial and boundary conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \ \text{ for } \ x > 0, \qquad \text{while } \ u(0,t) = h(t) \ \text{ for } \ t > 0$$

can now be solved by simply adding the solutions from the two special cases (7) (8) just treated.

Exercise: For the semi-infinite string $0 < x$, solve the initial-boundary value problem where the end at $x = 0$ is free (Neumann boundary condition):

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \ \text{ for } \ x > 0, \qquad \text{while } \ u_x(0, t) = 0 \ \text{ for } \ t > 0.$$

Finite string: $0 < x < L$
In the case of a finite string, such as a violin string, one must evidently also say something about the motion of the end points $x = 0$ and $x = L$. One typical situation is where we specify the position of these boundary points:

$$\text{left end:} \quad u(0, t) = \varphi(t), \qquad \text{right end:} \quad u(L, t) = \psi(t). \tag{9}$$

Thus, if the ends are tied down we would let $f(t) = g(t) = 0$. The equations (9) are called *boundary conditions*. As an alternate, one can impose other similar boundary conditions. Thus, if the right end is allowed to move freely and the left end is fixed ($\varphi(t) = \psi(t) = 0$), then the above boundary conditions become

$$u(0, t) = 0 \qquad \frac{\partial u}{\partial x}(L, t) = 0, \tag{10}$$

The condition at $x = L$ asserts the slope is zero there (that the slope at a free end is zero follows from physical considerations not given here).
There is no simple "closed form" solution of the mixed initial-boundary value problem (1),(4), (9), even in the case $f(t) = g(t) = 0$. The standard procedure one uses is *separation of variables* (see section **??** below). The solution is found as a Fourier series.

Conservation of Energy For both physical and mathematical reasons, it is important to consider the energy in a vibrating string. Here we work with an infinite string.

$$E(t) = \tfrac{1}{2} \int_{-\infty}^{\infty} (u_t^2 + c^2 u_x^2)\, dx \tag{11}$$

The term u_t^2 is for the kinetic energy and $c^2 u_x^2$ the potential energy. (Here we have assumed the mass density is 1; otherwise $E(t)$ should be multiplied by that constant.) For this integral to converge, we need to assume that u_t and u_x decay fast enough at $\pm\infty$. From the d'Alembert formula (5), this follows if the initial conditions decay at infinity.
We prove *energy is conserved* by showing that $dE/dt = 0$. This is a straightforward computation involving one integration by parts — in which the boundary terms don't appear because of the decay of the solution at infinity.

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} (u_t u_{tt} + c^2 u_x u_{xt})\, dx = \int_{-\infty}^{\infty} u_t (u_{tt} - c^2 u_{xx})\, dx = 0, \tag{12}$$

where in the last step we used the fact that u is a solution of the wave equation.

Exercises

1. For a finite string $0 < x < L$ with zero boundary conditions: $u(0,t) = u(L,t) = 0$, define the energy as
$$E(t) = \tfrac{1}{2}\int_0^L (u_t^2 + c^2 u_x^2)\,dx. \tag{13}$$
Show that energy is conserved. Show that energy is also conserved if one uses the free boundary condition $\partial u/\partial x = 0$ at either — or both — endpoints.

2. For a finite string $0 < x < L$ let u be a solution of the modified wave equation
$$u_{tt} + b(x,t)u_t = u_{xx} + a(x,t)u_x \tag{14}$$
with zero Dirichlet boundary conditions: $u(0,t) = u(L,t) = 0$, where we assume that $|a(x,t)|,\ |b(x,t)| < M$ for some constant M. Define the energy by (13).
   a) Show that $E(t) \le e^{\alpha t}E(0)$ for some constant α depending only on M. [SUGGESTION: Use the inequality $2ab \le a^2 + b^2$.]
   b) What happens if you replace the Dirichlet boundary conditions by the Neumann boundary condition $\nabla u \cdot N = 0$ on the boundary (ends) of the string?
   c) Generalize part a) to a bounded region Ω in $\mathbb{R}^n$.

Two and three space dimensions
In higher space dimensions, the wave equation is $u_{tt} = c^2\Delta u$. Thus, in two and three space dimensions
$$u_{tt} = c^2(u_{xx} + u_{yy}) \qquad \text{and} \qquad u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}). \tag{15}$$
Two dimensional waves on a drum head and waves on the surface of a lake are described by the first equation while sound and light waves are described by the second. Just as in the one dimensional case we can prescribe the initial position and initial velocity of the solution. For instance, in two space variables
$$\text{initial position} \qquad u(x,y,0) = f(x,y) \tag{16}$$
$$\text{initial velocity} \qquad u_t(x,y,0) = g(x,y). \tag{17}$$

Formulas for the solution in $\mathbb{R}^2$ and $\mathbb{R}^3$
There are standard formulas for the solution of the *initial value problem* (the term *Cauchy problem* is often called).

TECHNICAL OBSERVATION Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Say we want to solve

$$u_{tt} = \Delta u, \qquad \text{with} \quad u(\mathbf{x}, 0) = f(\mathbf{x}) \quad \text{and} \quad u_t(\mathbf{x}, 0) = g(\mathbf{x}). \tag{18}$$

Let $v(\mathbf{x}, t)$ and $w(\mathbf{x}, t)$, respectively, be the solutions of

$$v_{tt} = \Delta v, \qquad \text{with} \quad v(\mathbf{x}, 0) = 0 \quad \text{and} \quad v_t(\mathbf{x}, 0) = f(\mathbf{x}). \tag{19}$$

and

$$w_{tt} = \Delta w, \qquad \text{with} \quad w(\mathbf{x}, 0) = 0 \quad \text{and} \quad w_t(\mathbf{x}, 0) = g(\mathbf{x}). \tag{20}$$

Then v_t also satisfies the wave equation but with initial conditions $v_t(\mathbf{x}, 0) = f(\mathbf{x})$ and $v_{tt} = 0$. Thus the solution of (18) is $u(\mathbf{x}, t) = v_t(\mathbf{x}, t) + w(\mathbf{x}, t)$. Since both (19) and (20) have zero initial position, one can find $u(\mathbf{x}, t)$ after solving only problems like (20). This is utilized to obtain the following two formulas.

Let $\mathbf{x} = (x, y)$. For the two (space) dimensional wave equation it is

$$u(x, y, t) = \frac{1}{2\pi c} \frac{\partial}{\partial t} \left(\iint\limits_{r \le ct} \frac{f(\xi, \eta)}{\sqrt{c^2 t^2 - r^2}} \, d\xi \, d\eta \right) + \frac{1}{2\pi c} \iint\limits_{r \le ct} \frac{g(\xi, \eta)}{\sqrt{c^2 t^2 - r^2}} \, d\xi \, d\eta, \tag{21}$$

where $r^2 = (x - \xi)^2 + (y - \eta)^2$.

In three (space) dimensions, $\mathbf{x} = (x, y, z))$ one has

$$u(x, y, z, t) = \frac{\partial}{\partial t} \left(\frac{1}{4\pi c^2 t} \iint\limits_{r = ct} f(\xi, \eta, \zeta) \, dA \right) + \frac{1}{4\pi c^2 t} \iint\limits_{r = ct} g(\xi, \eta, \zeta) \, dA, \tag{22}$$

where $r^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2$ and dA is the element of surface area on the sphere centered at (x, y, z) with radius $r = ct$.

These are called *Kirchoff's formulas*. It is simplest first to obtain the formula in the three space dimensional case (22), and then obtain the two dimensional case (21) from the special three dimensional case where the initial data $f(x, y, z)$ and $g(x, y, z)$ are independent of z. This observation is called *Hadamard's method of descent*.

Exercises

1. Maxwell's equations for an electromagnetic field $E(x, t) = (E_1, E_2, E_3)$, $B(x, t) = (B_1, B_2, B_3)$ in a vacuum are

$$E_t = \operatorname{curl} B, \quad B_t = -\operatorname{curl} E, \quad \operatorname{div} B = 0, \quad \operatorname{div} E = 0.$$

   Show that each of the components E_j and B_j satisfy the wave equation $u_{tt} = u_{xx}$. Also, show that if initially $\operatorname{div} B(x, 0) = 0$ and $\operatorname{div} E(x, 0) = 0$, then $\operatorname{div} B(x, t) = 0$ and $\operatorname{div} E(x, t) = 0$ for all $t > 0$.

2. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and consider the equation

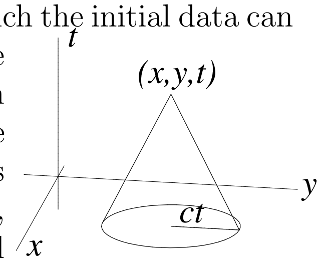
$$\frac{\partial^2 u}{\partial t^2} = \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k},$$

where the coefficients a_{jk} are constants and (without loss of generality — why?) $a_{kj} = a_{jk}$. If the matrix $A = (a_{jk})$ is positive definite, show there is a change of variable $x = Sy$, where S is an $n \times n$ invertible matrix, so that in these new coordinates the equation becomes the standard wave equation

$$\frac{\partial^2 u}{\partial t^2} = \sum_{\ell=1}^{n} \frac{\partial^2 u}{\partial^2 y_\ell}.$$

Domain of dependence and finite signal speed

As before, it is instructive to examine intersection of the domain of dependence with the plane $t = 0$, in other words, to determine the points x for which the initial data can influence the signal at a later time. In the two dimensional case (21), the intersection of the domain of dependence of the solution at (x_0, y_0, t_0) with the plane $t = 0$ is the entire disc $r \leq ct_0$, while in the three dimensional case (22), the domain of dependence is only the sphere $r = ct_0$, *not* the solid ball $r \leq ct_0$. Physically, this is interpreted to mean that two dimensional waves travel with a maximum speed c, but may move slower, while three dimensional waves always propagate with the exact speed c.



This difference in observed in daily life. If one drops a pebble into a calm pond, the waves (ripples) move outward from the center but ripples persist even after the initial wave has passed. On the other hand, an analogous light wave, such as a flash of light, moves outward as a sharply defined signal and does not persist after the initial wave has passed. Consequently, it is quite easy to transmit high fidelity waves in three dimensions — but not in two. Imagine the problems in attempting to communicate using something like Morse code with waves on the surface of a pond.

For the two space variable wave equation, the *characteristics* are the surfaces of all light cones $(x - \xi)^2 + (y - \eta)^2 = c^2 t^2$. In three space dimensions, the characteristics are the three dimensional light cones. They are the hypersurfaces in space-time with $(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 = c^2 t^2$.

Energy and Causality

One can also give a different prove of results concerning the domain of dependence using an *energy method*. This technique is especially useful in more general situations where explicit formulas such as (21)–(22) are not available.

Let $x = (x_1, \ldots, x_n)$ and let $u(x,t)$ be a smooth solution of the n-dimensional wave equation

$$u_{tt} = c^2 \Delta u \qquad \text{where} \qquad \Delta u = u_{x_1 x_1} + \cdots + u_{x_n x_n}, \tag{23}$$

with initial data

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x) \tag{24}$$

(Physicists often write the Laplacian, Δ, as ∇^2. Some mathematicians define Δ with a minus sign, so for them, in $\mathbb{R}^1$, $\Delta u = -u''$. Thus, one must be vigilant about the sign convention.)

Conservation of energy
Just as in the one dimensional case, we use the energy

$$E(t) = \tfrac{1}{2} \int_{\mathbb{R}^n} (u_t^2 + c^2 |\nabla u|^2)\, dx,$$

where we assume the solution is so small at infinity that this integral (and those below) converges. To prove conservation of energy, we show that $dE/dt = 0$, The computation is essentially identical to the one dimensional case we did above, only here we replace the integration by parts by the divergence theorem.

$$\frac{dE}{dt} = \int_{\mathbb{R}^n} (u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t)\, dx = \int_{\mathbb{R}^n} u_t (u_{tt} - c^2 \Delta u)\, dx = 0.$$

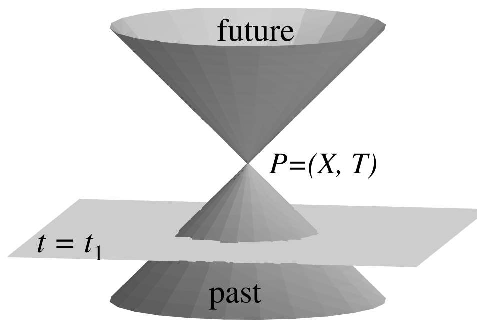
An immediate consequence of this is the uniqueness result: the wave equation (23) with initial conditions (24) has at most one solution. For if there were two solutions, v and w, then $u := v - w$ would be a solution of the wave equation with zero initial data, and hence zero initial energy. Since energy, $E(t)$, is conserved, $E(t) = 0$ for all time $t \geq 0$. Because the integrand in E is a sum of squares, then $u_t = 0$ and $\nabla u = 0$ for all $t \geq 0$. Thus $u(x,t) \equiv$const.. However $u(x,0) = 0$ so this constant can only be zero.
In two and three space dimensions this uniqueness also follows from the explicit formulas (21)–(22). However, the approach using energy also works when there are no explicit formulas.

Causality — using energy

Energy gives another approach to determine the domains of dependence and influence of the wave equation. Let $P = (X, T)$ be a point in space-time and let

$\mathcal{K}_P = \{(x, t) \colon \|x - X\| \le c|t - T|\}$

be the *light cone* with apex P. This cone has two parts, that with $t > T$ is the *future light cone* while that with $t < T$ is the *past light cone*. In the two and three (space) dimensional case, from the explicit formulas for the solution we have seen that the value of the solution at P only depends on points in the past light cone, and can only influence the solution at points in the future light cone. Here we give another demonstration of this that does not rely on the earlier explicit formulas.



First, say $t_1 < T$ and let $D(t_1)$ be the intersection of $\mathcal{K}_P$ with the plane $t = t_1$. Define the "energy" function as

$$E(t) = \tfrac{1}{2} \int_{D(t)} (u_t^2 + c^2 |\nabla u|^2) \, dx.$$

Theorem —enphIf $u(x, t)$ is a solution of the wave equation, and if $t_1 < t_2 < T$, then

$$\tfrac{1}{2} \int_{D(t_2)} (u_t^2 + c^2 |\nabla u|^2) \, dx \le \tfrac{1}{2} \int_{D(t_1)} (u_t^2 + c^2 |\nabla u|^2) \, dx,$$

that is, energy $E(t)$ is non-increasing for $t \le T$.
Consequently, if for some $t_1 < T$ we have $u(x, t_1) = 0$ and $u_t(x, t_1) = 0$ for all $x \in D(t_1)$, then $u(x, t) = 0$ for all points in the cone with $t_1 \le t \le T$.

Proof We will show that $dE(t)/dt \le 0$ for $0 \le t \le T$. In $\mathbb{R}^n$ we use spherical coordinates centered at X, we have $dx = dr \, d\omega_r$, where $d\omega_r$ is the element of "area" on the $n - 1$ sphere of radius r. Since the radius of the ball $D(t)$ is $c(T - t)$ we find that

$$E(t) = \tfrac{1}{2} \int_0^{c(T-t)} \left(\int_{S(r)} \left(u_t^2 + c^2 |\nabla u|^2 \right) \, d\omega_r \right) \, dr,$$

where $S(r)$ is the $n - 1$ sphere (in the plane) with radius r and centered at (X, t). Hence

$$\frac{dE}{dt} = \int_{D(t)} \left(u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t \right) \, dx - \tfrac{c}{2} \int_{S(c(T-t)))} \left(u_t^2 + c^2 |\nabla u|^2 \right) \, d\omega_{c(T-t)}.$$

Note that the integral on the right is just

$$\int_{\partial D(t)} \left(u_t^2 + c^2|\nabla u|^2\right)\, dA,$$

where dA is the element of "area" on $\partial D(t)$. Since

$$\nabla u \cdot \nabla u_t = \nabla \cdot (u_t \nabla u) - u_t \Delta u,$$

then by the divergence theorem we have

$$\int_{\partial D(t)} \nabla u \cdot \nabla u_t = \int_{\partial D(t)} u_t (\nabla u \cdot \nu)\, dA - \int_{D(t)} u_t \Delta u\, dx,$$

where ν is the unit outer normal vector to $\partial D(t)$. Upon substituting into the formula for dE/dt, we find

$$E'(t) = \int_{D(t)} u_t(u_{tt} - c^2 \Delta u)\, dx + \tfrac{c}{2} \int_{\partial D(t)} \left[2cu_t \nabla u \cdot \nu - \left(u_t^2 + c^2|\nabla u|^2\right)\right]\, dA.$$

Next, we note that $u_{tt} = c^2 \Delta u$ so the first integral is zero. For the second term we use the standard inequality $2ab \le a^2 + b^2$ for any real a, b to obtain the estimate

$$|2cu_t \nabla u \cdot \nu| \le +2c|u_t \nabla u| \le u_t^2 + c^2 |\nabla u|^2.$$

Consequently, $E'(t) \le 0$. This completes the proof.

There are two immediate consequences of the energy inequality of this theorem.

Corollary [uniqueness]. There is at most one solution of the inhomogeneous wave equation $u_{tt} - c^2 \Delta u = f(x,t)$ with initial data (24).

Corollary [domain of influence]. *Let u be a solution of the initial value problem (23)–(24). If $u(x,0)$ and $u_t(x,0)$ are zero outside the ball $\{\|x - X\| < \rho\}$, then for $t > 0$, the solution $u(x,t)$ is zero outside the forward light cone $\{\|x - X\| < \rho + ct\}$, $t > 0$.*

Thus, for $t > 0$, the domain of influence of the ball $\{\|x - X\| < \rho\}$ is contained in the cone $\{\|x - X\| < \rho + ct\}$ in the sense that if one changes in the initial data only in this ball, then the solution can change only in the cone.

Exercise: Let $u(x,t)$ be a solution of the wave equation (14) for $x \in \mathbb{R}$. Use an energy argument to show that the solution u has the same domain of dependence and range of influence as in the special case where $a(x,t) = b(x,t) = 0$.

Variational Characterization of the Lowest Eigenvalue
The formula (**??**) is essentially identical to the formula $\lambda = \langle x,\, Ax\rangle/\|x\|^2$ for the eigenvalues of a self-adjoint matrix A. A standard fact in linear algebra is that the lowest eigenvalue is given by $\lambda_1 = \min_{x\neq 0}\langle x,\, Ax\rangle/\|x\|^2$ (proof?). It is thus natural to surmise that the lowest eigenvalue of the Laplacian satisfies

$$\lambda_1 = \min \frac{\displaystyle\int_\Omega |\nabla\varphi|^2\,dx}{\displaystyle\int_\Omega |\varphi|^2\,dx}, \tag{25}$$

where the minimum is taken over all C^1 functions that satisfy the Dirichlet boundary condition $\varphi = 0$ on $\partial\Omega$. Assuming there is a function $\varphi \in C^2(\Omega)) \cap C^1(\overline{\Omega})$ that minimizes (**??**), we will show that it is an eigenfunction with lowest eigenvalue λ_1. To see this say such a φ minimizes the functional

$$J(v) = \frac{\displaystyle\int_\Omega |\nabla v|^2\,dx}{\displaystyle\int_\Omega |v|^2\,dx},$$

so $J(\varphi) \le J(v)$ for all $v \in C^1(\Omega)$ with $v = 0$ on $\partial\Omega$. Let $F(t) := J(\varphi + th)$ for any $h \in C^1(\Omega)$ with $h = 0$ on $\partial\Omega$ and all real t. Then $F(t)$ has its minimum at $t = 0$ so by elementary calculus, $F'(0) = 0$. By a straightforward computation, just as in the case of matrices,

$$F'(0) = 2\frac{\displaystyle\int_\Omega (\nabla\varphi\cdot\nabla h - \lambda_1\varphi h)\,dx}{\displaystyle\int_\Omega |\varphi|^2\,dx}.$$

We integrate the first term in the numerator by parts. There are no boundary terms since $h = 0$ on $\partial\Omega$. Thus

$$F'(0) = 2\frac{\displaystyle\int_\Omega [(-\Delta\varphi - \lambda_1\varphi)h]\,dx}{\displaystyle\int_\Omega |\varphi|^2\,dx}.$$

Since $F'(0) = 0$ for *all* of our h, we conclude the desired result:

$$-\Delta\varphi = \lambda_1\varphi.$$

Equation (25) is called the *variational characterization of the lowest eigenvalue*. There are analogous formulas for higher eigenvalues. Such formulas useful for computing numerical approximations to eigenvalues, and also to prove the existence of eigenvalues and eigenfunctions. The fraction in (25) is called the *Raleigh* (or *Raleigh-Ritz*) *quotient*.

Equation (25) implies the *Poincaré inequality*

$$\int_\Omega |\varphi|^2 \leq c(\Omega) \int_\Omega |\nabla\varphi|^2 \, dx \tag{26}$$

for all $\varphi \in C^1(\Omega)$ that vanish on $\partial\Omega$ (these are our admissible φ). Moreover, it asserts that $1/\lambda_1(\Omega)$ is the best value for the constant c.

It is instructive to give a direct proof of the Poincar'e inequality since it will give an estimate for the eigenvalue $\lambda_1(\Omega)$. Let V be a vector field on $\mathbb{R}^n$ (to be chosen later). For any of our admissible φ, by the divergence theorem

$$0 = \int_{\partial\Omega} \varphi^2 V \cdot N \, dA = \int_\Omega \operatorname{div}(\varphi^2 V)\, dx = \int_\Omega [\varphi^2 \operatorname{div} V + \varphi\nabla\varphi\cdot V]\, dx,$$

where N is the unit outer normal vector field on $\partial\Omega$. Now pick V so that div $= 1$, say $V = (x_1 - \alpha, 0, \ldots, 0)$. Then picking the constant α appropriately, $|V| \leq w/2$, where w is the width of Ω in the x_1 direction. Therefore, by the Schwarz inequality,

$$\int_\Omega [\varphi^2 \, dx \leq \frac{w}{2}\left[\int_\Omega \varphi^2\, dx\right]^{1/2}\left[\int_\Omega |\nabla\varphi|^2\, dx\right]^{1/2}.$$

Squaring both sides and canceling gives (26) with $c = (w/2)^2$, so $\lambda_1(\Omega) \geq w^2/4$.

Using the variational characterization(25), it is easy to prove a physically intuitive fact about vibrating membranes: *larger membranes have a lower fundamental frequency*. To prove this, say $\Omega \subset \Omega_+$ are bounded domains with corresponding lowest eigenvalues $\lambda_1(\Omega)$ and $\lambda_1(\Omega_+)$. Both of these eigenvalues are minima of the functional (25), the only difference being the class of functions for which the minimum is taken. Now every admissible function for the smaller domain Ω is zero on $\partial\Omega$ and hence can be extended to the larger domain by setting it to be zero outside Ω. It is now also an admissible function for the larger domain Ω_+. Therefore, for the larger domain the class of admissible functions for $J(v)$ is larger than for the smaller domain Ω. Hence its minimum $\lambda_1(\Omega_+)$ is no larger than $\lambda(\Omega)$.
Using similar reasoning, one can prove a number of related facts, and also get explicit estimates for eigenvalues. For instance, if we place $\Omega \subset \mathbb{R}^2$ in a rectangle Ω_+, since using Fourier series we can compute the eigenvalues for a rectangle, we get a lower bound for $\lambda(\Omega)$.

The inhomogeneous equation. Duhamel's principle.
There are also formulas for the solution of the inhomogeneous wave equation

$$Lu := u_{tt} - c^2 \Delta u = F(x,t). \tag{27}$$

The approach is analogous to Lagrange's method of variation of parameters, which gives a formula for the solution of an inhomogeneous equation such as $u'' + u = F(t)$ in terms of solutions of the homogeneous equation. The method is called *Duhamel's principle.*

We illustrate it for the wave equation, seeking a solution of (27) with initial conditions

$$u(x,0) = 0 \quad u_t(x,0) = 0.$$

Since we are solving a differential equation, it is plausible to find a solution as an integral in the form

$$u(x,t) = \int_0^t v(x,t;s)\, ds \tag{28}$$

where the function $v(x,t;s)$, which depends on a parameter s, is to be found. This clearly already satisfies the initial condition $u(x,0) = 0$. Working formally, we have

$$u_t(x,t) = \int_0^t v_t(x,t;s)\, ds + v(x,t;t),$$

so $u_t(x,0) = 0$ implies $v(x,0;0) = 0$. In fact, we will further restrict v by requiring that $v(x,t;t) = 0$ for all $t \ge 0$. Then the formula for u_t simplifies and

$$u_{tt}(x,t) = \int_0^t v_{tt}(x,t;s)\, ds + v_t(x,t;s)\big|_{s=t}.$$

The similar formula for Δu is obvious. Substituting these into the wave equation (27) we want

$$F(x,t) = Lu(x,t) = \int_0^t Lv(x,t;s)\, ds + v_t(x,t;s)\big|_{s=t}.$$

This is evidently satisfied if $Lv = 0$ and $v_t(x,t;s)\big|_{s=t} = F(x,t)$ along with $v(x,t;t) = 0$ for all $t \ge 0$.

Because the coefficients in the wave equation do not depend on t, our results can be simplified a bit by writing $v(x,t;s) = w(x,t-s;s)$ so for each fixed s, the function $w(x,t;s)$ satisfies

$$w_{tt} = c^2 \Delta w \qquad \text{with } \; w(x,0,;s) = 0 \; \text{ and } \; w_t(x,0;s) = F(x,s). \tag{29}$$

We can now find w by using our earlier formulas. For instance, in three space variables, from (22)

$$w(x,t;s) = \frac{1}{4\pi c^2 t} \iint\limits_{\|\xi - x\| = ct} F(\xi, s)\, dA_\xi,$$

where dA_ξ is the element of surface area on the sphere centered at x with radius ct, that is, $\|\xi - x\| = ct$. Therefore from (28)

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi c^2} \int_0^t \frac{1}{t-s} \iint\limits_{\|\xi - x\| = c(t-s)} F(\xi, s)\, dA_\xi\, ds \\ &= \frac{1}{4\pi c} \int_0^t \iint\limits_{\|\xi - x\| = c(t-s)} \frac{F(\xi, t - \|\xi - x\|/c)}{\|\xi - x\|}\, dA_\xi\, ds. \end{aligned}$$

But in spherical coordinates, the element of volume $d\xi = c dA_\xi ds$ so we finally obtain

$$u(x,t) = \frac{1}{4\pi c^2} \iiint\limits_{\|\xi - x\| \le ct} \frac{F(\xi, t - \|\xi - x\|/c)}{\|\xi - x\|}\, d\xi. \tag{30}$$

Thus, to solve the inhomogeneous equation we integrate over backward cone $\|\xi - x\| \le ct$, which is exactly the domain of dependence of the point (x,t).

Exercises

1. Use Duhamel's principle to obtain a formula for the solution of

   $$-u'' + k^2 u = f(x), \quad x \in \mathbb{R}, \quad \text{with} \quad u(0) = 0,\ u'(0) = 0.$$

   Similarly, do this for $-u'' - k^2 u = f(x)$.

2. Use (21) to derive the analog of (30) for one and two space variables.

3. Let $x \in \mathbb{R}^n$.

   a) If function $w(x)$ depends only on the distance to the origin, $r = \|x\|$, show that

   $$\Delta u = \frac{\partial^2 u}{\partial^2 r} + \frac{n-1}{r}\frac{\partial u}{\partial r}.$$

   b) Investigate solutions $u(x,t)$, $x \in \mathbb{R}^3$ of the wave equation $u_{tt} = \Delta u$ where $u(x,t) = v(r,t)$ depends only on r and t. For instance, let $v(r,t) := rw(r,t)$

and note that v satisfies a simpler equation. Use this to solve the wave equation in $\mathbb{R}^3$ where the initial data are radial functions:

$$v(r,0) = \varphi(r), \qquad v_t(r,0) = \psi(r).$$

[SUGGESTION: Extend both φ and ψ as even functions of r.]
Are there solutions of $w_t t(r,t) = \Delta(r,t)$ with the form $w(r,t) = h(r)g(r-t)$?

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