Math 241 **Final Exam** Jerry L. Kazdan

May 11, 2015 12:00 – 2:00

DIRECTIONS This exam has two parts. Part A has 5 shorter question (8 points each so total 40 points), Part B has 5 traditional problems (15 points each, so total is 75 points). Maximum score is thus $40 + 75 = 115$ points.

Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Please remember to silence your cellphone before the exam and keep it out of sight for the duration of the test period. This exam begins promptly at 12:00 and ends at 2:00; anyone who continues working after time is called may be denied the right to submit his or her exam or may be subject to other grading penalties. Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

Part A 5 Shorter Problems, 8 points each.

A–1. Find a function $u(x,t)$ that satisfies $u_t - u = 7x$ with $u(x,0) = 0$.

SOLUTION: A particular solution of the inhomogeneous equation is $u_p = -7x$. The general solution of the homogeneous equation is $u_0 = k(x)e^t$ for any function $k(x)$. Thus the general solution of the inhomogeneous equation is

$$u(x,t) = k(x)e^t - 7x.$$

To satisfy the initial condition we need

$$0 = u(x,0) = k(x) - 7x$$

so $k(x) = 7x$ and $u(x,t) = 7xe^t - 7x$.

A–2. Let $\Omega \subset \mathbb{R}^2$ be a bounded region with boundary $\partial\Omega$.

Say $u(x,y,t)$ is a solution of $u_t = \Delta u + 2u + e^t \sin(x + 2y)$ in Ω with

$u(x,y,t) = 0$ on $\partial\Omega$ and $u(x,y,0) = 0$;

and $v(x,y,t)$ is a solution of $v_t = \Delta v + 2v$ in Ω with

$v(x,y,t) = x^2 y$ on $\partial\Omega$ and $v(x,y,0) = \cos(2x)$.

Find a function $w(x,y,t)$ that satisfies $w_t = \Delta w + 2w + 3e^t \sin(x + 2y)$ in Ω with

$w(x,y,t) = 5x^2 y$ on $\partial\Omega$ and $w(x,y,0) = 5\cos(2x)$.

[Your solution should give a formula for w in terms of u and v].

SOLUTION: By linearity, $w(x,y,t) = 3u(x,y,t) + 5v(x,y,t)$.

A–3. Say $u(x,t)$ is a solution of the wave equation $u_{tt} = 9u_{xx}$ for all $-\infty < x < \infty$, with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

Find all the points on the x-axis that can influence the solution at $x = 2$, $t = 4$.

SOLUTION: The d'Alembert formula gives

$$u(x,t) = \tfrac{1}{2}[f(x+3t) + f(x-3t)] + \tfrac{1}{2\cdot 3}\int_{x-3t}^{x+3t} g(s)\,ds.$$

Letting $x = 2$ and $t = 4$ we find that the only points on the x-axis that can influence the solution are in the interval $-10 \le x \le 14$.

A–4. Let $u(x,t)$ be a solution of

$$u_t = u_{xx} + au$$

where a is a constant. Show that by making a change of variable $u(x,t) = \varphi(t)v(x,t)$, for an astute choice of $\varphi(t)$ the function v satisfies the simpler heat equation $v_t = v_{xx}$.

SOLUTION: By a computation, $u_t = \varphi v_t + \varphi_t v$ so $u_t = u_{xx} + au$ becomes

$$\varphi v_t + \varphi_t v = \varphi v_{xx} + a\varphi v,$$

that is

$$\varphi v_t = \varphi v_{xx} + [-\varphi_t + a\varphi]v.$$

To kill the last term, just let $\varphi(t) = e^{at}$. This idea also works if a is a function of t, in which case let $\varphi(t) = e^{\int a(t)\,dt}$.

A–5. Suppose $u(x,y)$ satisfies the Laplace equation $\Delta u = 0$ on the square $0 < x < 1$, $0 < y < 1$ and has boundary values $+1$ on the top and bottom ($y = 0, 1$) and boundary values -1 on the left and right ($x = 0, 1$). What is the value of u at the point $(\frac{1}{3}, \frac{1}{3})$?

[HINT: What happens to the PDE and boundary conditions if the square is reflected about the line $y = x$? Say $v(x,y)$ is the solution of the reflected problem. How are u and v related? What can you say about the *sum* of u and its reflection, v?]

SOLUTION: Here $v(x,y) = u(y,x)$ Then $v_{xx} + v_{yy} = u_{yy} + u_{xx} = 0$ so v also satisfies the Laplace equation. Let $w(x,y) = u(x,y) + v(x,y)$. Then w satisfies the Laplace equation and is zero on the boundary. Thus $w(x,y) = 0$ throughout the square. Consequently, $u(y,x) = -u(x,y)$ so on the diagonal $u(x,x) = 0$ for all x in $0 < x < 1$ (u is discontinuous at the corners of the square).

Part B 5 standard problems (15 points each, so 75 points)

B–1. Suppose $f(x) = \begin{cases} -1 & \text{for} \quad -\pi \le x \le 0 \\ \;\;1 & \text{for} \quad \;\;\; 0 < x \le \pi \end{cases}$.

a) Compute the Fourier series of $f(x)$ for the interval $-\pi \le x \le \pi$.

SOLUTION: The Fourier series is

$$f(x) \sim a_0 + \sum_{k=1}^{\infty} \left[a_k \cos kx + b_k \sin kx\right], \tag{1}$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \quad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, dx, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx\, dx.$$

Because f is odd, the $a_k = 0$. Also,

$$b_k = \frac{2}{\pi} \int_0^{\pi} \sin kx\, dx = -\frac{2}{k\pi} \cos kx \Big|_0^{\pi} = \begin{cases} 0 & k \text{ even} \\ \frac{4}{k\pi} & k \text{ odd} \end{cases}$$

Thus,

$$f(x) \sim \frac{4}{\pi} \left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots \right].$$

b) Draw a graph of the Fourier series (computed above) for $-2\pi \le x \le 2\pi$ and put an "X" at all points of discontinuity.

SOLUTION: `https://upload.wikimedia.org/wikipedia/commons/thumb/2/2c/Fourier_Series.svg/`
There are jump discontinuities at $x = k\pi,\ k = 0, \pm 1, \pm 2, \ldots$.

B–2. Let Ω be the unit disk in the plane $\mathbb{R}^2$ and note that it is contained in the square with vertices at $(-1. -1)$, $(1, -1)$, $(1, 1)$, and $(-1, 1)$.

Let $-\Delta v_1 = \lambda_1 v_1$, where λ_1 is the lowest eigenvalue of the Laplacian in Ω and $v_1(x, y)$ is the corresponding eigenfunction with $v_1(x, y) = 0$ on $\partial\Omega$.

Find a number $m > 0$ so that $\lambda_1 > m$.

SOLUTION: From the Rayleigh-Ritz quotient, if $\lambda_1(\Omega)$ is the lowest eigenvalue of the Laplacian with zero Dirichlet boundary condition and if $\Omega \subset \Omega_+$, then $\lambda_1(\Omega_+) < \lambda_1(\Omega)$.

Apply this where Ω_+ is the square with corners at $(\pm 1, \pm 1)$ and $(\pm 1, \mp 1)$. Recall (or compute!) that for a rectangle in the plane with sides a and b then the eigenvalues of the Laplacian are

$$\lambda_{k,\ell} = \left(\frac{k\pi}{a}\right)^2 + \left(\frac{\ell\pi}{b}\right)^2, \quad \ell = 1, 2, 3, \ldots$$

so for our square ($a = b = 2$), $\lambda_1(\Omega) > \lambda_1(\Omega_+) = 2\frac{\pi^2}{4}$.

B–3. Let T be the triangle $0 \le x \le 1$, and $0 \le y \le x$. You may use without proof that the Dirichlet eigenfunctions of the Laplacian on T are given by

$$v_{nk}(x,y) = \sin n\pi x \sin k\pi y - \sin k\pi x \sin n\pi y, \quad k = 2,3,\ldots \text{ and } n = 1,2,\ldots,k-1. \tag{2}$$

and that these eigenfunctions are orthogonal and satisfy

$$\iint_T |v_{nk}|^2 \, dA = \frac{1}{4} \tag{3}$$

Use this to write down a formula for the general series solution of the wave equation $u_{tt} = \Delta u$ in the triangle T with $u = 0$ on the boundary, ∂T. As part of this, include the formulas you would use to find the coefficients in the above series in terms of the initial position and velocity.

SOLUTION: Using separation of variables, seek the special solutions $u(x,y,t) = v(x,y)T(t)$. Then

$$\ddot{T}(t) + \lambda T(t) = 0 \quad \text{and} \quad \Delta v + \lambda v = 0,$$

where λ is a constant and $v(x,y) = 0$ on the boundary, ∂T. By the Raleigh-Ritz quotient, $\lambda > 0$. Using the explicit formulas (2) we find $\lambda_{nk} = (n^2 + k^2)\pi^2$. Then the general solution of the wave equation in T is

$$u(x,y,t) = \sum_{k=2}^{\infty} \left(\sum_{n=1}^{k-1} \left[A_{kn} \cos \sqrt{\lambda_{kn}}\, t + B_{kn} \sin \sqrt{\lambda_{kn}}\, t \right] v_{kn}(x,y) \right).$$

Note that

$$u(x,y,0) = \sum_{k=2}^{\infty} \left(\sum_{n=1}^{k-1} A_{kn} v_{kn}(x,y) \right) \quad \text{and} \quad u_t(x,y,0) = \sum_{k=2}^{\infty} \left(\sum_{n=1}^{k-1} B_{kn} \sqrt{\lambda_{kn}}\, v_{kn}(x,y) \right)$$

Because the v_{kn} are orthogonal with normalization (3), the coefficients are determined by the formulas

$$A_{kn} = 4\langle u(x,y,0),\, v_{kn}\rangle \quad \text{and} \quad B_{kn}\sqrt{\lambda_{kn}} = 4\langle u_t(x,y,0),\, v_{kn}\rangle.$$

B–4. Let $\Omega \subset \mathbb{R}^3$ be a bounded region and let $u(x,y,z,t)$ be a solution of

$$u_{tt} - \Delta u + u = 0 \quad \text{in } \Omega \quad \text{with} \quad u = 0 \quad \text{on } \partial\Omega.$$

Define the "Energy" $E(t)$ by

$$E(t) = \tfrac{1}{2} \iiint_{\Omega} \left[u_t^2 + |\nabla u|^2 + u^2 \right] \, d\,\text{Vol}.$$

a) Show that $E(t)$ is a constant.

SOLUTION:
$$\frac{dE}{dt} = \iiint_\Omega [u_t u_{tt} + \nabla u \cdot \nabla u_t + u u_t]\, dVol.$$

By Green's theorem, since $u = 0$ on $\partial\Omega$, then

$$\iiint_\Omega \nabla u \cdot \nabla u_t \, dVol = -\iint_\Omega u_t \Delta u \, dVol.$$

Therefore
$$\frac{dE}{dt} = \iiint_\Omega u_t [u_{tt} - \Delta u + u] \, dVol = 0.$$

so $E(t)$ is a constant.

b) Use this to obtain a "uniqueness result" for the solution of this equation with specified initial position, $u(x, y, z, 0)$, and velocity, $u_t(x, y, z, 0)$.

SOLUTION: Say u and v are both solutions of this partial differential equation in Ω with the *same* initial conditions

$$u(x, y, z, 0) = v(x, y, z, 0) \quad \text{and} \quad u_t(x, y, z, 0) = v_t(x, y, z, 0)$$

and also $u = v$ on the boundary $\partial\Omega$. Let $w = u - v$. Then w satisfies $w_{tt} - \Delta w + w = 0$ in Ω, $w(x, y, z, 0) = 0$, $w_t(x, y, z, 0)$ and $w = 0$ on $\partial\Omega$. By part a), the energy, $E(t)$, associated with w is a constant. Using the initial conditions, this constant is zero. Because $E(t)$ is a sum of non-negative terms, $w(x, y, z, t) = 0$. Consequently $u = v$.

B–5. a) In the square $0 \le x \le a$, $0 \le y \le a$ in the plane, a substance is diffusing whose molecules multiply at a rate proportional to the concentration. It thus satisfies

$$u_t = \Delta u + \gamma u, \tag{4}$$

where γ is a constant. Assume that $u = 0$ on all four sides of the square. What is the condition on γ so that the concentration $u(x, y, t) \to 0$ as $t \to \infty$?

b) Generalize part a) to a bounded region Ω in $\mathbb{R}^2$ with boundary condition $u = 0$ on $\partial\Omega$. Your solution should relate γ to the Dirichlet eigenvalues of the Laplacian.

SOLUTION: We work immediately in a general bounded region Ω and use separation of variables, seeking special solutions of equation (4) in the form $u(x, y, t) = v(x, y)T(t)$ where $v(x, y) = 0$ on the boundary, $\partial\Omega$. Then

$$\frac{\ddot{T}(t)}{T} - \gamma = \frac{\Delta v}{v} = -\lambda,$$

where, using the Rayleigh quotient, $\lambda > 0$. As usual,

$$\ddot{T} + (\lambda - \gamma)T = 0 \qquad \text{and} \qquad \Delta v + \lambda v = 0.$$

The functions $v(x,y) = v_n(x,y)$ are the eigenfunctions of the Laplacian in Ω with boundary condition $v_n = 0$ on $\partial\Omega$. The corresponding eigenvalues are $\lambda_n > 0$. For *all* of the solutions of $\ddot{T} + (\lambda_n - \gamma)T = 0$ to decay as $t \to \infty$ we need that $\lambda_n - \gamma > 0$, that is, $\gamma < \lambda_1$. In the particular case where Ω is a square with sides a this means $\gamma < 2(\pi/a)^2$. Then the general solution of equation (4)is

$$u(x,y,t) = \sum_{n=1}^{\infty} A_n e^{-\sqrt{\lambda_n - \gamma}\, t} v_n(x,y)$$

where the coefficients A_n are usually determined from the initial condition, $u(x,y,0)$.

REMARK: Problem A-4 (above) gives another approach to this.