My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this exam.

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Signature PRINTED NAME

Math 241 **Final Exam** Jerry L. Kazdan
May 11, 2015 12:00 – 2:00

DIRECTIONS This exam has two parts. Part A has 5 shorter question (8 points each so total 40 points), Part B has 5 traditional problems (15 points each, so total is 75 points). Maximum score is thus $40 + 75 = 115$ points.

Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Please remember to silence your cellphone before the exam and keep it out of sight for the duration of the test period. This exam begins promptly at 12:00 and ends at 2:00; anyone who continues working after time is called may be denied the right to submit his or her exam or may be subject to other grading penalties. Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

Part A 5 Shorter Problems, 8 points each.

Score	
A-1	
A-2	
A-3	
A-4	
A-5	
B-1	
B-2	
B-3	
B-4	
B-5	
Total	

A–1. Find a function $u(x,t)$ that satisfies $u_t - u = 7x$ with $u(x,0) = 0$.

A–2. Let $\Omega \subset \mathbb{R}^2$ be a bounded region with boundary $\partial\Omega$.

Say $u(x, y, t)$ is a solution of $u_t = \Delta u + 2u + e^t \sin(x + 2y)$ in Ω with

$$u(x, y, t) = 0 \text{ on } \partial\Omega \text{ and } u(x, y, 0) = 0;$$

and $v(x, y, t)$ is a solution of $v_t = \Delta v + 2v$ in Ω with

$$v(x, y, t) = x^2 y \text{ on } \partial\Omega \text{ and } v(x, y, 0) = \cos(2x).$$

Find a function $w(x, y, t)$ that satisfies $w_t = \Delta w + 2w + 3e^t \sin(x + 2y)$ in Ω with

$$w(x, y, t) = 5x^2 y \text{ on } \partial\Omega \text{ and } w(x, y, 0) = 5\cos(2x).$$

[Your solution should give a formula for w in terms of u and v].

A–3. Say $u(x, t)$ is a solution of the wave equation $u_{tt} = 9u_{xx}$ for all $-\infty < x < \infty$, with $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$.

Find all the points on the x-axis that can influence the solution at $x = 2$, $t = 4$.

A–4. Let $u(x,t)$ be a solution of

$$u_t = u_{xx} + au$$

where a is a constant. Show that by making a change of variable $u(x,t) = \varphi(t)v(x,t)$, for an astute choice of $\varphi(t)$ the function v satisfies the simpler heat equation $v_t = v_{xx}$.

A–5. Suppose $u(x,y)$ satisfies the Laplace equation $\Delta u = 0$ on the square $0 < x < 1$, $0 < y < 1$ and has boundary values $+1$ on the top and bottom $(y = 0, 1)$ and boundary values -1 on the left and right $(x = 0, 1)$. What is the value of u at the point $(\frac{1}{3}, \frac{1}{3})$?

[HINT: What happens to the PDE and boundary conditions if the square is reflected about the line $y = x$? Say $v(x,y)$ is the solution of the reflected problem. How are u and v related? What can you say about the *sum* of u and its reflection, v?]

Part B 5 standard problems (15 points each, so 75 points)

B–1. Suppose $f(x) = \begin{cases} -1 & \text{for} \quad -\pi \le x \le 0 \\ 1 & \text{for} \quad 0 < x \le \pi \end{cases}$.

a) Compute the Fourier series of $f(x)$ for the interval $-\pi \le x \le \pi$.

b) Draw a graph of the Fourier series (computed above) for $-2\pi \le x \le 2\pi$ and put an "X" at all points of discontinuity.

B–2. Let Ω be the unit disk in the plane $\mathbb{R}^2$ and note that it is contained in the square with vertices at $(-1. -1)$, $(1, -1)$, $(1,\ 1)$, and $(-1,\ 1)$.

Let $-\Delta v_1 = \lambda_1 v_1$, where λ_1 is the lowest eigenvalue of the Laplacian in Ω and $v_1(x, y)$ is the corresponding eigenfunction with $v_1(x, y) = 0$ on $\partial\Omega$.

Find a number $m > 0$ so that $\lambda_1 > m$.

B–3. Let T be the triangle $0 \leq x \leq 1$, and $0 \leq y \leq x$. You may use without proof that the Dirichlet eigenfunctions of the Laplacian on T are given by

$$v_{nk}(x, y) = \sin n\pi x \sin k\pi y - \sin k\pi x \sin n\pi y, \quad k = 2, 3, \ldots \text{ and } n = 1, 2, \ldots, k-1.$$

and that these eigenfunctions are orthogonal and satisfy

$$\iint_T |v_{nk}|^2 \, dA = \frac{1}{4}$$

Use this to write down a formula for the general series solution of the wave equation $u_{tt} = \Delta u$ in the triangle T with $u = 0$ on the boundary, ∂T. As part of this, include the formulas you would use to find the coefficients in the above series in terms of the initial position and velocity.

B–4. Let $\Omega \subset \mathbb{R}^3$ be a bounded region and let $u(x, y, z, t)$ be a solution of

$$u_{tt} - \Delta u + u = 0 \quad \text{in } \Omega \quad \text{with} \quad u = 0 \quad \text{on } \partial\Omega.$$

Define the "Energy" $E(t)$ by

$$E(t) = \frac{1}{2} \iiint_{\Omega} \left[u_t^2 + |\nabla u|^2 + u^2 \right] \, d\,\text{Vol}.$$

a) Show that $E(t)$ is a constant.

b) Use this to obtain a "uniqueness result" for the solution of this equation with specified initial position, $u(x, y, z, 0)$, and velocity, $u_t(x, y, z, 0)$.

B–5. a) In the square $0 \leq x \leq a$, $0 \leq y \leq a$ in the plane, a substance is diffusing whose molecules multiply at a rate proportional to the concentration. It thus satisfies

$$u_t = \Delta u + \gamma u,$$

where γ is a constant. Assume that $u = 0$ on all four sides of the square. What is the condition on γ so that the concentration $u(x, y, t) \to 0$ as $t \to \infty$?

b) Generalize part a) to a bounded region Ω in $\mathbb{R}^2$ with boundary condition $u = 0$ on $\partial\Omega$. Your solution should relate γ to the Dirichlet eigenvalues of the Laplacian.