Math 241 | Exam 2 | Jerry Kazdan
March 24, 2015 | | 10:30 – 11:50

DIRECTIONS This exam has two parts. Part A has 4 shorter problems (8 points each, so 32 points) while Part has 4 standard problems, (15 points each so total 60 points). Maximum total score is thus 92 points.
Closed book, no calculators or computers– but you may use one 3” $\times$ 5” card with notes on both side.

Remember to silence your cellphone before the exam and keep it *out of sight* for the duration of the test period. This exam begins promptly at 10:30 and ends at 11:500pm; anyone who continues working after time is called may be denied the right to submit his or her exam or may be subject to other grading penalties. Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

PART A: 4 problems, 8 points each (32 points total)

A–1. Write e^x as the sum of an even function and an odd function.

A–2. Say for some function φ the function $u(x,t) = \varphi(x - ct)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$ for $-\infty < x < \infty$. If $u(x,0) = e^x$, what is the initial velocity?

A–3. Say $u(x,t)$ is a solution of the wave equation $u_{tt} = 4u_{xx}$ for all $-\infty < x < \infty$ with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Find the largest interval $a \le x \le b$ where modifying f and g inside this interval can change the value of $u(3,5)$.

A–4. Let $f(x) = \begin{cases} 0 & \text{for } \ 0 \le x \le \pi/2 \\ 1 & \text{for } \ \pi/2 < x \le \pi \end{cases}$.

If you extend f to $-\pi \le x \le \pi$ as an odd function and let $g(x)$ be the Fourier series of the resulting 2π periodic function, what can you say about the convergence of the Fourier series of $g(x)$ for $= \pi \le x \le \pi$? [Describe the convergence at *every* point of $-\pi \le x \le \pi$].

PART B 4 traditional problems. 15 points each (so 60 points).

B–1. For $0 < x < 1$ and $t > 0$ let $u(x,t)$ be a solution of the wave equation

$$u_{tt} = u_{xx} \qquad \text{with} \qquad u_x(0,t) = u_x(1,t) = 0.$$

Define the energy as $E(t) = \displaystyle\int_0^1 [(u_t)^2 + (u_x)^2]\, dx$.
Show that

$$E(t) = E(0) \quad \text{for all } t \ge 0.$$

B–2. Consider the plane $\mathbb{R}^2$ in polar coordinates.

a) Find a constant c so that $\varphi(r) = cr^2$ is a solution of $\Delta\varphi = 1$.

b) For any constants a and b , find a function $v(r)$ that satisfies the Laplace equation $\Delta v = 0$ in the annulus $1 \leq r \leq 2$ with $v(1) = a$ and $v(2) = b$.

c) Find a function $u(r)$ in the annulus that satisfies $\Delta u = 1$ with $u(1) = 0$ and $u(2) = 0$.

B–3. Let $f(x)$ be a smooth real 2π periodic function and we seek a 2π periodic solution $u(x)$ of $u'' + 4u = f$. Say the (complex) Fourier series of $f(x) = \sum_{-\infty}^{\infty} a_n e^{inx}$ and the Fourier series of the desired solution is $u(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$.

a) Find a formula for the c_n in terms of the a_n.

b) Show that a necessary condition for the solution u to exist is that f satisfy

$$\int_{-\pi}^{\pi} f(x)e^{2ix}\,dx = 0 \qquad \text{and} \qquad \int_{-\pi}^{\pi} f(x)e^{-2ix}\,dx = 0.$$

B–4. Consider the following eigenvalue problem for $v(x)$:

$$xv'' + 2v' + \lambda x^3 v = 0 \qquad \text{for} \qquad 1 < x < 2.$$

a) Find a function $h(x)$ so that after multiplying the above equation by $h(x)$ it has the standard Sturm-Liouville form $(pv')' + qv + \lambda\sigma v = 0$ (so find h, p, q, and σ in terms of the coefficients in the original equation).

b) Use this to show that with the boundary conditions $v(1) = 0$ and $v'(2) = 0$ all the eigenvalues satis fy $\lambda \geq 0$.