My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this exam.

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Signature PRINTED NAME

Math 241 **Exam 1** Jerry Kazdan

Feb. 17, 2015 10:30 – 11:50

DIRECTIONS This exam has two parts. Part A has 5 shorter problems (8 points each, so 40 points) while Part has 4 standard problems, (15 points each so total 60 points). Maximum total score is thus 100 points.

Closed book, no calculators or computers– but you may use one 8.5" × 11" sheet of paper notes on ONE side.

Remember to silence your cellphone before the exam and keep it *out of sight* for the duration of the test period. This exam begins promptly at 10:30 and ends at 11:500pm; anyone who continues working after time is called may be denied the right to submit his or her exam or may be subject to other grading penalties. Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

PART A: 5 problems, 8 points each (40 points total)

Score	
A-1	
A-2	
A-3	
A-4	
A-5	
B-1	
B-2	
B-3	
B-4	
Total	

A–1. The general series solution of the heat equation $u_t = u_{xx}$ for $0 < x < 1$, $t > 0$ with homogeneous Neumann boundary conditions $u_x(0,t) = u_x(1,t) = 0$ is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty}(A_n \cos n\pi x)e^{-(n\pi)^2 t}.$$

a) Show that $\lim_{t\to\infty} u(x,t) = \text{constant}$.

b) If the initial condition is $u(x,0) = f(x)$, find a formula for this constant in terms of f.

A–2. Find a function $u(x,t)$ that satisfies

$$u_t - u = 0 \qquad \text{with initial condition} \quad u(x,0) = 2 + x^2.$$

A–3. a) Show that the function $u(x,y) = x^4 - 6x^2y^2 + y^4$ is a solution of the two-dimensional Laplace equation $\nabla^2 u = 0$.

b) By the mean-value property, what is the average of u on the circle of radius 5 centered at the point (1, 1)?

A–4. Let $u(x,t)$ be a solution of the heat equation $u_t = u_{xx} + \alpha^2 u$ where $\alpha > 0$ is a constant.

a) Regardless of any initial or boundary conditions, what are the possible equilibrium solutions?

b) If this equation describes the temperature in a rod, $0 < x < L$ (with a heat generation term $\alpha^2 u$), and if the rod is insulated at both ends ($u_x = 0$ there), then what are the possible equilibrium solutions? What condition(s) must α satisfy?

A–5. Suppose that u satisfies the heat equation $u_t = u_{xx}$ in a rod $(0 < x < L)$ with the mixed boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = A(t), \quad u(L,t) = B(t).$$

Pick the appropriate function u_0 below so that the function $v(x,t) := u(x,t) - u_0(x,t)$ satisfies the *homogeneous* boundary conditions $v_x(0,t) = 0$ and $v(L,t) = 0$.

$$(a) \quad u_0(x,t) = xA(t) + \tfrac{x^2}{2L}[B(t) - A(t)]$$
$$(b) \quad u_0(x,t) = B(t) + A(t)[x - L]$$
$$(c) \quad u_0(x,t) = A(t) + \tfrac{x}{L}[B(t) - A(t)]$$

Briefly show your work. No credit for guessing.

Part B begins on the next page

PART B 4 traditional problems. 15 points each (so 60 points).

B–1. Let $u(x,t)$ be the temperature in a rod, $0 < x < 2$ the satisfies the initial and boundary value problem

$$\begin{aligned} &u_t = u_{xx} + cx && \text{for } \ 0 < x < 2, \quad t > 0 \\ &u_x(0,t) = 0 && \text{and } \ u_x(2,t) = -2, \\ &u(x,0) = \frac{8}{\pi}\cos\frac{\pi}{4}x, \end{aligned}$$

where c is a constant. Denote the thermal energy in the rod by $H(t) = \displaystyle\int_0^2 u(x,t)\,dx$;

a) What is the physical meaning of the boundary condition $u_x(0,t) = 0$?

b) Compute $\dfrac{dH}{dt}$ (it will involve the constant c).

c) Use this to compute $H(t)$.

d) For which value(s) of c does the limit $v(x)\lim_{t\to\infty} u(x,t)$ exist, and what is this limit?

B–2. Consider the following boundary value problem for a heat conduction in a rod:

$$u_t + 2u = 4u_{xx} \quad \text{for} \quad 0 < x < 1$$
$$u(0,t) = 0, \quad \text{and} \quad u(1,t) = 0.$$

a) Use separation of variables to determine all product solutions of this problem.

b) Find the solution for which the initial temperature is $u(x,0) = 2\sin \pi x + \sin 3\pi x$.

B–3. Find the harmonic function $u(r, \theta)$ (so $\nabla^2 u = 0$) on the unit disk, $r < 1$ that satisfies the boundary condition

$$u(1, \theta) = 2 + 3 \sin 4\theta.$$

B–4. Use the method of separation of variables to find a full infinite series solution u(x, t) of the wave equation

$$u_{tt} = u_{xx}$$

for x in the interval $[0, 1]$ and time $t > 0$ subject to the mixed boundary conditions $u(0, t) = 0$ and $u_x(1, t) = 0$ [this applies to sound waves in a clarinet, since one of its ends is open].

You are *not* asked to compute the coefficients in terms of specific initial data.

[Last revised February 20, 2015]