My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this exam.

______________________________ ______________________________

Signature PRINTED NAME

Math 241
Feb. 17, 2015

Exam 1

Jerry Kazdan
10:30 – 11:50

DIRECTIONS This exam has two parts. Part A has 5 shorter problems (8 points each, so 40 points) while Part has 4 standard problems, (15 points each so total 60 points). Maximum total score is thus 100 points.
Closed book, no calculators or computers– but you may use one 8.5" × 11" sheet of paper notes on ONE side.

Remember to silence your cellphone before the exam and keep it *out of sight* for the duration of the test period. This exam begins promptly at 10:30 and ends at 11:500pm; anyone who continues working after time is called may be denied the right to submit his or her exam or may be subject to other grading penalties. Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

PART A: 5 problems, 8 points each (40 points total)

Score	
A-1	
A-2	
A-3	
A-4	
A-5	
B-1	
B-2	
B-3	
B-4	
Total	

A–1. The general series solution of the heat equation $u_t = u_{xx}$ for $0 < x < 1$, $t > 0$ with homogeneous Neumann boundary conditions $u_x(0,t) = u_x(1,t) = 0$ is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty}(A_n \cos n\pi x)e^{-(n\pi)^2 t}.$$

a) Show that $\lim_{t\to\infty} u(x,t) = \text{constant}$.

b) If the initial condition is $u(x,0) = f(x)$, find a formula for this constant in terms of f.

A–2. Find a function $u(x,t)$ that satisfies

$$u_t - u = 0 \qquad \text{with initial condition} \quad u(x,0) = 2 + x^2.$$

A–3. a) Show that the function $u(x,y) = x^4 - 6x^2y^2 + y^4$ is a solution of the two-dimensional Laplace equation $\nabla^2 u = 0$.

b) By the mean-value property, what is the average of u on the circle of radius 5 centered at the point (1, 1)?

A–4. Let $u(x,t)$ be a solution of the heat equation $u_t = u_{xx} + \alpha^2 u$ where $\alpha > 0$ is a constant.

a) Regardless of any initial or boundary conditions, what are the possible equilibrium solutions?

b) If this equation describes the temperature in a rod, $0 < x < L$ (with a heat generation term $\alpha^2 u$), and if the rod is insulated at both ends ($u_x = 0$ there), then what are the possible equilibrium solutions? What condition(s) must α satisfy?

A–5. Suppose that u satisfies the heat equation $u_t = u_{xx}$ in a rod $(0 < x < L)$ with the mixed boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = A(t), \quad u(L,t) = B(t).$$

Pick the appropriate function u_0 below so that the function $v(x,t) := u(x,t) - u_0(x,t)$ satisfies the *homogeneous* boundary conditions $v_x(0,t) = 0$ and $v(L,t) = 0$.

(a) $u_0(x,t) = xA(t) + \frac{x^2}{2L}[B(t) - A(t)]$
(b) $u_0(x,t) = B(t) + A(t)[x - L]$
(c) $u_0(x,t) = A(t) + \frac{x}{L}[B(t) - A(t)]$

Briefly show your work. No credit for guessing.

PART B BEGINS ON THE NEXT PAGE

PART B 4 traditional problems. 15 points each (so 60 points).

B–1. Let $u(x,t)$ be the temperature in a rod, $0 < x < 2$ the satisfies the initial and boundary value problem

$$\begin{aligned} &u_t = u_{xx} + cx && \text{for } \ 0 < x < 2, \quad t > 0 \\ &u_x(0,t) = 0 && \text{and } \ u_x(2,t) = -2, \\ &u(x,0) = \frac{8}{\pi}\cos\frac{\pi}{4}x, && \end{aligned}$$

where c is a constant. Denote the thermal energy in the rod by $H(t) = \int_0^2 u(x,t)\,dx$;

a) What is the physical meaning of the boundary condition $u_x(0,t) = 0$?

b) Compute $\dfrac{dH}{dt}$ (it will involve the constant c).

c) Use this to compute $H(t)$.

d) For which value(s) of c does the limit $v(x)\lim_{t\to\infty} u(x,t)$ exist, and what is this limit?

B–2. Consider the following boundary value problem for a heat conduction in a rod:

$$u_t + 2u = 4u_{xx} \quad \text{for} \quad 0 < x < 1$$
$$u(0,t) = 0, \quad \text{and} \quad u(1,t) = 0.$$

a) Use separation of variables to determine all product solutions of this problem.

b) Find the solution for which the initial temperature is $u(x,0) = 2\sin \pi x + \sin 3\pi x$.

B–3. Find the harmonic function $u(r, \theta)$ (so $\nabla^2 u = 0$) on the unit disk, $r < 1$ that satisfies the boundary condition

$$u(1, \theta) = 2 + 3 \sin 4\theta.$$

B–4. Use the method of separation of variables to find a full infinite series solution u(x, t) of the wave equation

$$u_{tt} = u_{xx}$$

for x in the interval $[0, 1]$ and time $t > 0$ subject to the mixed boundary conditions $u(0, t) = 0$ and $u_x(1, t) = 0$ [this applies to sound waves in a clarinet, since one of its ends is open].

You are *not* asked to compute the coefficients in terms of specific initial data.

Supplementary Problems These were almost on the exam.

C–1. This problem concerns using separation of variables to find special solutions of

$$u_t - u_x = 0.$$

Assume $u(x,t)$ has the special form $u(x,t) = \phi(x)G(t)$ Find ODEs for $\phi(x)$ and $G(t)$. [NOTE: You are not being asked to solve these ODE's.]

C–2. **Fall 2012 Final** Suppose the concentration $u(x,t)$ of a chemical satisfies Ficks law: $\phi = -k\frac{\partial u}{\partial x}$. Assume there is no source to generate chemicals in the region $0 \le x \le 1$ and that cross-section area $A(x) = 1$. Answer the following questions under the assumption that initial concentration is $u(x,0) = x(1-x)$ and the flow of chemical at both ends is specified by the boundary conditions $-k\frac{\partial u}{\partial x}(0,t) = a$ and $-k\frac{\partial u}{\partial x}u(1,t) = b$.

a) Determine the rate of change of the total amount of chemical in the region.

b) Determine the total amount of chemical in the region as a function of time.

c) Under what condition is there (eventually) an equilibrium distribution of the chemical – and what is it?

C–3. Suppose that u satisfies the heat equation in a rod ($0 < x < L$) with the boundary conditions on the left. Match the boundary condition on the left with the reference temperature u_0 on the right so that $v = u - u_0$ satisfies homogeneous boundary conditions.

i)	$\frac{\partial u}{\partial x}(0,t) = A(t),\ u(L,t) = B(t)$	*(a)*	$u_0(x,t) = xA(t) + \frac{x^2}{2L}[B(t) - A(t)]$
ii)	$u(0,t) = A(t),\ u(L,t) = B(t)$	*(b)*	$u_0(x,t) = B(t) + A(t)[x - L]$
iii)	$\frac{\partial u}{\partial x}(0,t) = A(t),\ \frac{\partial u}{\partial x}(L,t) = B(t)$	*(c)*	$u_0(x,t) = A(t) + \frac{x}{L}[B(t) - A(t)]$

C–4. The PDE $u_t = u_{xx} - u$ governs the temperature distribution $u(x,t)$ of a rod, say $0 < x < 1$, that has some heat loss through its lateral sides; Assume the boundary conditions

$$u_x(0,t) = u_x(1,t) = 0, \quad \text{and} \quad \int_0^1 u(x,0)\,dx = 1.$$

Compute the energy in the rod: $E(t) = \int_0^1 u(x,t)\,dx$. [HINT: Calculate $E'(t)$, use the PDE to simplify, and then solve the ODE for $E(t)$.]

C–5. Find the solution $u(x,t)$ of the PDE

$$u_{tt} + 2u_t = u_{xx}$$

for x in the interval $[-1, 1]$ subject to periodic boundary conditions and also the initial conditions

$$u(x, 0) = 0, \qquad u_t(x, 0) = \cos \pi x + 3 \sin 3\pi x.$$

Your coefficients should be fully simplified.

C–6. Let L be the differential operator defined by the formula

$$Lu = u_t - u_{xx}$$

a) Show that the function $u_0 = \frac{1}{2}x(x - 1)$ is a particlular solution of $Lu_0 = -1$

b) Use this – and linearity – to describe how you would solve $Lu = -1$ on $0 < x < 1$ for all $t > 0$ subject to the initial condition $u(x, 0) = 0$ with the homogeneous Dirichlet boundary conditions $u(0, t) = u(1, t) = 0$. [NOTE: You are *not* being asked to actually find the solution.]

C–7. Solve the Laplace equation on the half disk $0 \le \theta \le \pi$, $0 \le r \le 1$, subject to the boundary conditions $u(1, \theta) = \theta + 2$ and $\frac{\partial u}{\partial \theta}(r, 0) = \frac{\partial u}{\partial \theta}(r, \pi) = 0$.

C–8. Consider the initial/boundary value problem for the temperature of a rod:

$$u_t = u_{xx} \qquad \text{for } \ 0 < x < 1, \quad t > 0$$

with

$$\text{boundary conditions} \quad u(0, t) = 0, \quad u_x(1, t) = 0,$$

and some initial condition, say $u(x, 0) = f(x)$. If $u(x, t)$ represents the temperature distribution of a heat conducting one dimensional rod, give a physical interpretation of these boundary conditions.

C–9. Write a series for the solution u of the modified heat equation

$$\frac{\partial u}{\partial t} + tu = k\nabla^2 u?$$

on the unit ball in three dimensions (and all $t \ge 0$) subject to homogeneous Dirichlet boundary conditions $u(x, y, z, t) = 0$ for all points (x, y, z) on the boundary of the ball and all $t \ge 0$. You need not relate the coefficients of the series to the initial data.

C–10. In the unit disk, $r^2 \le 1$ in the plane, consider the inhomogeneous Laplace equation

$$\nabla^2 u = r^2 \quad \text{with } u(1, \theta) = \cos 2\theta.$$

a) Find a particular solution $v(r)$ of this inhomogeneous equation with $v(1) = 0$. Note that you are seeking a v that does not depend on θ.

b) Find a solution $w(r, \theta)$ of the *homogeneous equation*

$$\nabla^2 w = r^2 \quad \text{with } w(1, \theta) = \cos 2\theta.$$

c) Use the above results to solve the original problem for $u(r, \theta)$.

C–11. Let $u(x, t)$ be the solution of the initial/boundary value problem for the heat equation: $u_t = u_{xx}$ for $0 < x < 1$, $t > 0$, $u(x, 0) = 0$, $u(0, t) = 0$, $u(1, t) = 1$.

a) Compute $v(x)$, where $v(x) = \lim_{t\to\infty} u(x, t)$ (assuming it exist).

b) Find $u(x, t)$. [HINT: First find $w(x, t) = u(x, t) - v(x)$].

c) Use the equilibrium solution and one term from the Fourier series of $u(x, t)$ to estimate how long T from $t = 0$ it will take for $u(\frac{1}{2}, t)$ to attain half its equilibrium value (in other words, estimate the value of T such that

$$u(\tfrac{1}{2}, t) < \tfrac{1}{2} \lim_{t\to\infty} u(\tfrac{1}{2}, t) \quad \text{for } t < T$$

and

$$u(\tfrac{1}{2}, t) > \tfrac{1}{2} \lim_{t\to\infty} u(\tfrac{1}{2}, t) \quad \text{for } t > T.$$

[Bonus points for explaining why this is a really good estimate.]