Square of Differences: Experimentation, Conjecture, and Proof

The Game

Mathematics involves experimentation and discovery, generalization and conjecture, and proof. The game *Square of Differences* supplies a valuable exercise in searching for understanding. The Proofs are simply the final report on your conclusions. The presentation below originated with a *New York Times* article
`http://wordplay.blogs.nytimes.com/2011/05/16/numberplay-dange`
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as presented by Lee Kennard.

One starts with a square whose four corners are labeled with nonnegative integers. One then computes the absolute value of the difference between the two endpoints of each edge, and reports that (nonnegative integer) on the corresponding midpoint.

Step 1. Draw a new square whose vertices are these four numbers, then repeat this process indefinitely. For example, if we denote the four original numbers by (1, 2, 3, 4), then the four differences are 1, 1, 1, and 3, which we report as (1, 1, 1, 3). (Order matters, but cyclic shifts do not.) This completes the first step;

Step 2. gives (0, 0, 2, 2),

Step 3. gives (0,2, 0, 2),

Step 4. gives (2, 2, 2, 2)

Step 5. finally gives (0, 0, 0,0).

We say that *a game ends* IF it reaches all zeros.

To the mathematician, a natural question to ask is whether the game always ends. That is, given four nonnegative integers as our starting point, is there some finite number of steps after which we are left with all zeros?

SHORT HOMEWORK:

Do experiments to see if some pattern emerges.

1. Play Square of Differences with (2, 3, 4, 5) and (10, 20, 30, 40) as your starting values. Choose three more starting values and play SoD for each.

2. In each row of each game, circle the largest number. What do you notice?

3. Adjust the rules to make a more primitive game that might be easier to understand: Copy your five games below, except replace all instances of even numbers with zeros and all instances of odd numbers with ones. What do you notice? [Polya's book *How To Solve It* has valuable suggestions for things to try when you are stuck on a question.]

4. Do you think SoD always ends? "I don't know." is an appropriate answer.) Explain.

[PAUSE TO EXPERIMENT AND THINK]

Square of Differences: 6 Experiments

Step 0	(1, 2, 3, 4)	(2, 3, 4, 5)	(10, 20, 30, 40)
Step 1	(1, 1, 1, 3)	(1, 1, 1, 3)	(10, 10, 10, 30)
Step 2	(0, 0, 2, 2)	(0, 0, 2, 2)	(0, 0, 20, 20)
Step 3	(0, 2, 0, 2)	(0, 2, 0, 2)	(0, 20, 0, 20)
Step 4	(2, 2, 2, 2)	(2, 2, 2, 2)	(20, 20, 20, 20)
Step 5	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)

Step 0	(1, 3, 5, 11)	(4, 3, 7, 11)	(4, 2, 6, 14)
Step 1	(2, 4, 6, 8)	(1, 4, 4, 7)	(2, 4, 8, 10)
Step 2	(2, 2, 2, 6)	(3, 0, 3, 6)	(2, 4, 2, 8)
Step 3	(0, 0, 4, 4)	(3, 3, 3, 3)	(2, 2, 6, 6)
Step 4	(0, 4, 0, 4)	(0, 0, 0, 0)	(0, 4, 0, 4)
Step 5	(4, 4, 4, 4)		(4, 4, 4, 4)
Step 6	(0, 0, 0, 0)		(0, 0, 0, 0)

Based on this, we make the following observations:

Monotonicity: At every step the maximum value either decreases or remains the same; the maximum value is never increased.

Add a constant: Comparing the first and second experiments, we see that if we add the same constant to each of the values, then after one step we are back where we started. Thus, we can always assume the minimum of the values is zero. I did not see how to use this.

Scaling: Comparing the first and third examples, if you multiply the starting values by any constant, then the remainder of the steps are multiplied by the same constant. This is clear.

MORAL: If you multiply the starting values by any constant, then if the game ends, it ends in the *same* number of steps. We'll say these two starting values are *equivalent*. For instance, the sixth experiment is equivalent (in this sense) to starting with (2, 1, 3, 7) whose maximum is half its original value, 14.

Even: It looks like after at most 4 steps, all the elements in the list are even . We prove this now.

Using the cyclic symmetry, there are only 6 possible starting configurations. After some experimentation (an important step when presenting your results to others), it is simplest to list them in the following order:

1. (even, even, even, even): already all even.
2. (odd, odd, odd, odd): 1 step to reduce to all even.
3. (odd, even, odd, even): 1 step reduces to the previous case.
4. (odd, even, even, odd): 1 step reduces to the previous case.
5. (even, odd, odd, odd) and 6. (odd, even, even, even):
   1 step reduces to the previous case.

Combining observations 1 and 3, we see that if the initial values are at most 1, then after at most 4 steps we get all zeroes and the game ends.

Always Ends: We now prove the game always ends in a finite number of steps.

However one begins, after at most 4 steps the values are always even, say $(2a,\ 2b,\ 2c,\ 2d)$ with maximum value at most the initial maximum. But this is equivalent to the game $(a,\ b,\ c,\ d)$ whose maximum is at most *half* that of the original game. Repeating this a finite number of times we eventually get a game with all even values $(0,\ 0\,0\,0)$, as desired.

We showed that the game must always end. How many steps before it does? Here is an example showing that for all n, there is a game that takes more that n steps. Define the sequence inductively by $s_1 = 0,\ s_2 = 0,\ s_3 = 1,\ s_n = s_{n-1} + s_{n-2} + s_{n-3}$, $n = 4,\ 5\,6\ldots$. Then a game starting with $s_n,\ s_{n+1},\ s_{n+2},\ s_{n+3}$ after 3 steps becomes $(s_{n-2},\ s_{n-1},\ s_n, s_{n+1}$. Thus a game starting with $(s_{2n},\ s_{2n+1},\ s_{2n+2},\ s + 2n + 3)$ will last $3n + 3$ steps. However, our proof shows that a game with maximum starting value N ends in at most $4 + 4\log_2(N)$ steps (this estimate can be improved) – but it may be much shorter, as illustrated by the initial values $(N. N,\ N,\ N)$ which takes only one step.

Variants. What if instead of a square one begins with a triangle or pentagon?