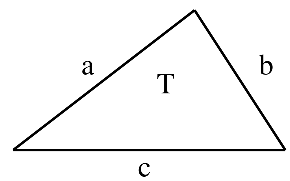
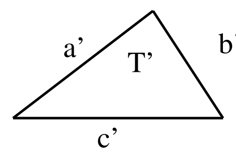
Pythagorean Theorem Using the Area of Similar Triangles

Let T be a right triangle whose sides have length a, b, and c (c is the hypotenuse). The Pythagorean Theorem says that

$$a^2 + b^2 = c^2. \tag{1}$$

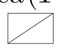


This is Euclid's second proof. It explicitly uses the area of similar right triangles. I am surprised that this classical approach, used briefly in [POLYA, p. 16-17], as well as much earlier, is not ued more often.



Let T' be a right triangle with sides a', b', and c' that is similar to T. Then the length of the corresponding sides of T and T' are proportional, that is, there is a *scaling factor* $t > 0$ so that

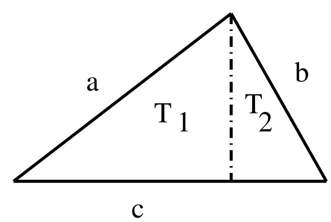
$$a' = ta, \quad b' = tb, \quad c' = tc.$$

First step: Compare Area(T) and Area(T').
A right triangle is half of a rectangle:  , so Area$(T) = \frac{1}{2}ab$ and

$$\text{Area}(T') = \tfrac{1}{2}a'b' = \tfrac{1}{2}(ta)(tb) = t^2\text{Area}(T).$$

Since $t = \frac{c'}{c}$, we find the simple formula

$$\text{Area}(T') = \left[\frac{\text{hypotenuse}(T')}{\text{hypotenuse}(T)}\right]^2 \text{Area}(T). \tag{2}$$



Now we use Euclid's key idea: For the right triangle T from the right angle drop a line perpendicular to the hypotenuse of T. This partitions T into two right triangles, T_1 and T_2. Both of them are similar to T because their corresponding angles are equal. By comparing the length of the hypotenuses of T, T_1 and T_2 we find the scaling factors:

	T	T_1	T_2
hypotenuse	c	a	b

By equation (2) Area$(T_1) = (a/c)^2$Area(T) and Area$(T_2) = (b/c)^2$Area(T). But Area$(T) =$ Area$(T_1) +$ Area(T_2) so

$$\text{Area}(T) = [(a/c)^2\,\text{Area}(T). + (b/c)^2]\,\text{Area}(T).$$

Dividing by Area(T) gives the Pythagorean formula (1).

Note that he importance of the Pythagorean formula is not for comparing areas, but, taking the square root of (1), for finding the distance between two points in an orthogonal coordinate system.

[POLYA] Polya, G. *Induction and Analogy in Mathematics*, Princeton University Press 1954.