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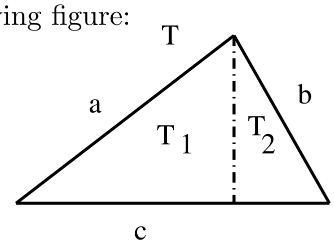
Pythagorean Theorem: A Simple Restatement

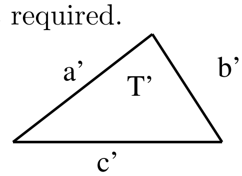
Euclid's Second Proof of the Pythagorean Theorem uses the following figure: Let T be a right triangle whose sides have length a, b, and c (c is the hypotenuse). Partition it into two triangles, T_1 and T_2 by drawing a line (the altitude) from the right angle perpendicular to the hypotenuse. We claim that the obvious formula



$$\text{Area}(T) = \text{Area}(T_1) + \text{Area}(T_2) \tag{1}$$

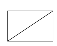
is exactly the Pythagorean Theorem – in a thin disguise.

To remove this disguise, first observe, following Euclid, that the right triangles T_1 and T_2 are both similar to T (they have the same angles), Thus we need to relate the areas of similar right triangles. As we shall see, this is straightforward. No cleverness required.



Let T' be a right triangle with sides a', b', and c' that is similar to T. Then the length of the corresponding sides of T and T' are proportional, that is, there is a *scaling factor* $t > 0$ so that

$$a' = ta, \quad b' = tb, \quad c' = tc.$$

A right triangle is half of a rectangle:  , so $\text{Area}(T) = \frac{1}{2}ab$ and

$$\text{Area}(T') = \tfrac{1}{2}a'b' = \tfrac{1}{2}(ta)(tb) = t^2\text{Area}(T).$$

Not surprising.[1] It will be helpful to use $t = \frac{c'}{c}$ to restate this as

$$\text{Area}(T') = \left[\frac{\text{hypotenuse}(T')}{\text{hypotenuse}(T)}\right]^2 \text{Area}(T). \tag{2}$$

Since for us $\text{hypotenuse}(T) = c$, $\text{hypotenuse}(T_1) = a$, and $\text{hypotenuse}(T_2) = b$, equation (2) implies

$$\text{Area}(T_1) = (a/c)^2\text{Area}(T) \quad \text{and} \quad \text{Area}(T_2) = (b/c)^2\text{Area}(T).$$

Thus, equation (1) states that

$$\text{Area}(T) = (a/c)^2\,\text{Area}(T) + (b/c)^2]\,\text{Area}(T).$$

Dividing by $\text{Area}(T)$ reveals the *Pythagorean formula* for the right triangle T

$$c^2 = a^2 + b^2 \tag{3}$$

Note that the importance of the Pythagorean formula is not for comparing areas, but, taking the square root of (3), for finding the distance between two points in an orthogonal coordinate system.

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[1]More generally: if $Q_1 \subset \mathbb{R}^2$ is any bounded "measurable" region and we rescale R^2 by $(x_1, x_2) \mapsto (tx_1, tx_2)$ with $t > 0$, then the rew Q_t has $\text{Area}(Q_t) = t^2\text{Area}(Q_1)$. The modification for R^n is obvious.