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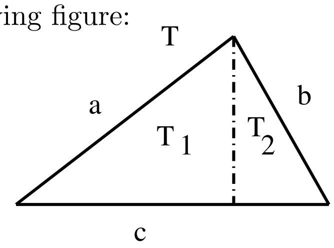
Pythagorean Theorem: A Simple Restatement

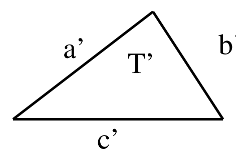
Euclid's Second Proof of the Pythagorean Theorem uses the following figure: Let T be a right triangle whose sides have length a, b, and c (c is the hypotenuse). Partition it into two triangles, T_1 and T_2 by drawing a line, the altitude, from the right angle that is perpendicular to the hypotenuse. We claim that the following obvious statement is the Pythagorean Theorem – in a thin disguise:



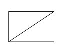
$$\text{Area}(T) = \text{Area}(T_1) + \text{Area}(T_2). \tag{1}$$

To remove this disguise, first observe that the right triangles T_1 and T_2 are both similar to T (they have the same angles), Thus we need to relate the areas of similar right triangles. As we shall see, this is straightforward. No cleverness required.

Let T' be a right triangle with sides a', b', and c' that is similar to T. Then the length of the corresponding sides of T and T' are proportional, that is, there is a *scaling factor* $t > 0$ so that



$$a' = ta, \quad b' = tb, \quad c' = tc.$$

A right triangle is half of a rectangle:  , so Area$(T) = \frac{1}{2}ab$ and

$$\text{Area}(T') = \tfrac{1}{2}a'b' = \tfrac{1}{2}(ta)(tb) = t^2\text{Area}(T).$$

Since $t = \frac{c'}{c}$, we find the desired simple geometric relationship:

$$\text{Area}(T') = \left[\frac{\text{hypotenuse}(T')}{\text{hypotenuse}(T)}\right]^2 \text{Area}(T). \tag{2}$$

Now apply this by comparing the length of the hypotenuses of T, T_1 and T_2 to find the scaling factors:

	T	T_1	T_2
hypotenuse	c	a	b

By equation (2), Area$(T_1) = (a/c)^2$Area(T) and Area$(T_2) = (b/c)^2$Area(T). Thus, equation (1) states that

$$\text{Area}(T) = (a/c)^2\,\text{Area}(T). + (b/c)^2]\,\text{Area}(T).$$

Dividing by Area(T) reveals the *Pythagorean formula* for the right triangle T.

$$c^2 = a^2 + b^2 \tag{3}$$

Note that the importance of the Pythagorean formula is not for comparing areas, but, taking the square root of (3), for finding the distance between two points in an orthogonal coordinate system.