work to have brought topology within the framework of rigorous mathematics, where intuition remains the source but not the final validation of truth. During this process, started by L. E. J. Brouwer, the significance of topology for almost the whole of mathematics has steadily increased. American mathematicians, in particular O. Veblen, J. W. Alexander, and S. Lefschetz, have made important contributions to the subject.

While topology is definitely a creation of the last hundred years, there were a few isolated earlier discoveries that later found their place in the modern systematic development. By far the most important of these is a formula, relating the numbers of vertices, edges, and faces of a simple polyhedron, observed as early as 1640 by Descartes, and rediscovered and used by Euler in 1752. The typical character of this relation as a topological theorem became apparent much later, after Poincaré had recognized "Euler's formula" and its generalizations as one of the central theorems of topology. So, for reasons both historical and intrinsic, we shall begin our discussion of topology with Euler's formula. Since the ideal of perfect rigor is neither necessary nor desirable during one's first steps in an unfamiliar field, we shall not hesitate from time to time to appeal to the reader's geometrical intuition.

§1. EULER'S FORMULA FOR POLYHEDRA

Although the study of polyhedra held a central place in Greek geometry, it remained for Descartes and Euler to discover the following fact: In a simple polyhedron let V denote the number of vertices, E the number of edges, and F the number of faces; then always

$$V - E + F = 2. \tag{1}$$

By a *polyhedron* is meant a solid whose surface consists of a number of polygonal faces. In the case of the regular solids, all the polygons are congruent and all the angles at vertices are equal. A polyhedron is *simple* if there are no "holes" in it, so that its surface can be deformed continuously into the surface of a sphere. Figure 120 shows a simple polyhedron which is not regular, while Figure 121 shows a polyhedron which is not simple.

The reader should check the fact that Euler's formula holds for the simple polyhedra of Figures 119 and 120, but does not hold for the polyhedron of Figure 121.

To prove Euler's formula, let us imagine the given simple polyhedron to be hollow, with a surface made of thin rubber. Then if we cut out

one of the faces of the hollow polyhedron, we can deform the remaining surface until it stretches out flat on a plane. Of course, the areas of the

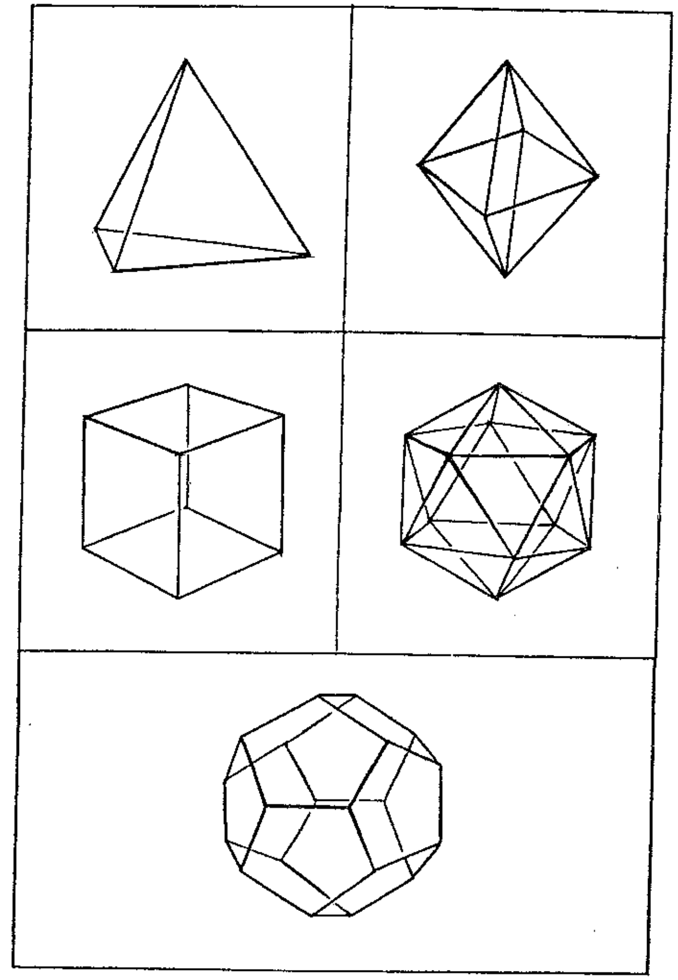

Fig. 119. The regular polyhedra.

faces and the angles between the edges of the polyhedron will have changed in this process. But the network of vertices and edges in the plane will contain the same number of vertices and edges as did the

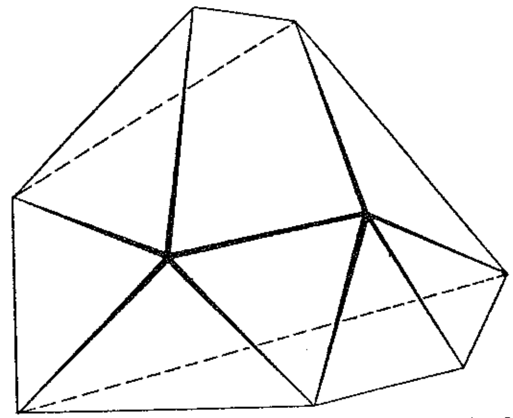

Fig. 120. A simple polyhedron. $V - E + F = 9 - 18 + 11 = 2$

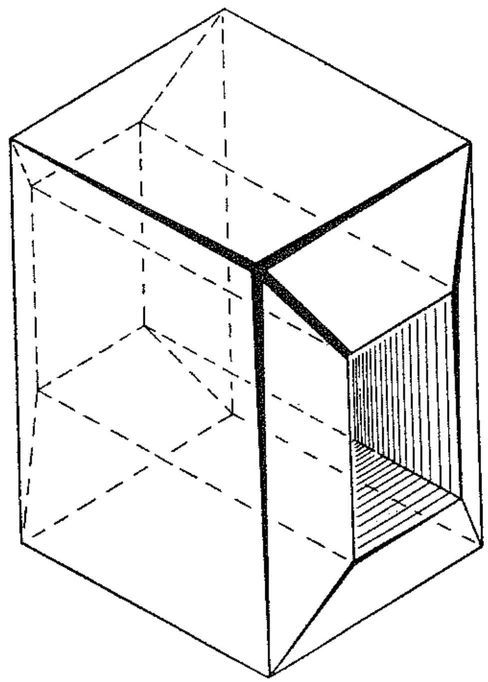

Fig. 121. A non-simple polyhedron. $V - E + F = 16 - 32 + 16 = 0$.

original polyhedron, while the number of polygons will be one less than in the original polyhedron, since one face was removed. We shall now show that for the plane network, $V - E + F = 1$, so that, if the removed face is counted, the result is $V - E + F = 2$ for the original polyhedron.

First we "triangulate" the plane network in the following way: In some polygon of the network which is not already a triangle we draw a diagonal. The effect of this is to increase both E and F by 1, thus preserving the value of $V - E + F$. We continue drawing diagonals joining pairs of points (Fig. 122) until the figure consists entirely of triangles, as it must eventually. In the triangulated network, $V - E + F$ has the value that it had before the division into tri-

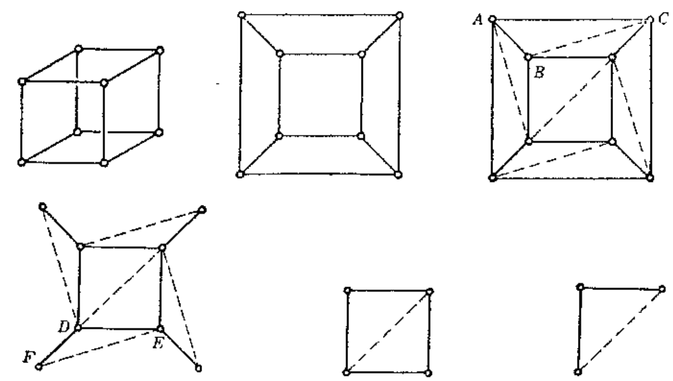

Fig. 122. Proof of Euler's theorem.

angles, since the drawing of diagonals has not changed it. Some of the triangles have edges on the boundary of the plane network. Of these some, such as ABC, have only one edge on the boundary, while other triangles may have two edges on the boundary. We take any boundary triangle and remove that part of it which does not also belong to some other triangle. Thus, from ABC we remove the edge AC and the face, leaving the vertices A, B, C and the two edges AB and BC; while from DEF we remove the face, the two edges DF and FE, and the vertex F. The removal of a triangle of type ABC decreases E and F by 1, while V is unaffected, so that $V - E + F$ remains the same. The removal of a triangle of type DEF decreases V by 1, E by 2, and F by 1, so that $V - E + F$ again remains the same. By a properly chosen sequence of these operations we can remove triangles with edges on the boundary (which changes with each removal), until finally only one triangle remains, with its three edges, three vertices, and one face. For this