

Bayes' Theorem

January 17, 2006

Over a million dollars worth of jewels have been stolen from the second floor of a mansion while the hostess was giving a grand dinner party below and there are only two cat burglars daring enough to have done the job – Freddy Fingers and Tommy Tiptoes. They have thumbed their noses at the police and have even sometimes been so bold as to leave calling cards but not this time. From the cards we know that Fingers has pulled off only 1/4 of the cat burglaries in recent years but 4/5 of his have involved over \$1 million in jewels. Tiptoes has done 4/5 of the cat burglaries but only half of his have netted more than \$1 million and half have netted less. There is news that the burglar has been caught but his name is still secret. A friend says, “I’ll give you 2 to 1 odds it was Tommy Tiptoes.” Should you take the bet?

Here is how we analyzed this problem in class. Imagine two urns, each containing a mixture of black and white balls, one labelled “Freddy Fingers” in which all the balls in addition have an “F” mark, and the other labelled “Tommy Tiptoes” with balls labelled “T”. Suppose that Fingers’ urn contains 80 black balls and 20 white, 100 all together; if you pulled a ball at random from this urn the probability that it was black would be the same as the fraction of Fingers’ burglaries that netted over \$1 million. In Tiptoes’ urn suppose that there were 150 black balls and 150 white, 300 all together; pulling a black ball from this urn would have the same probability as the fraction of Tiptoes’ burglaries that netted over \$1 million. Now combine the balls from the two urns into one large one (and mix thoroughly). Since Fingers’ urn contributed 100 balls and Tiptoes’ contributed 300, if you pulled a ball from this urn then the probability that it came from Fingers’ is the same as Fingers’ fraction of the cat burglaries, and the same for Tiptoes. But now we know that over \$1 million has been stolen and we are asking, in effect, with this additional information, what is the probability that Fingers did the job and what is the probability that Tiptoes did it. It is the same as if blindfolded we took a ball from the big urn and on seeing that it was in fact black, asked what is the probability that it came originally from Fingers’ urn or from Tiptoes’. Now there were a total of 230 black balls in the big urn, 80 coming from Fingers and 150 from Tiptoes, so the probability that the ball we took was one of Fingers’ is 80/230 and the probability that was one of Tiptoes’ is 150/230. The odds that it was one of Tiptoes is therefore 150/80 which is less than 2/1. You should take the bet.

In symbols, the problem is this. Let $p(F)$ be the *a priori* probability that

Fingers did the job, i.e., the probability before we have any other information, and $p(T)$ (which equals $1 - p(F)$) be the *a priori* probability that Tiptoes did it. We have some additional information – *conditional probabilities*– namely, the probability that it was a black ball (i.e. over a million dollars) *knowing that Fingers did it*, denoted $p(B|F)$ (read “probability of B given F ”), and similarly for Tiptoes. We are told that $p(F) = 1/4$, $p(T) = 3/4$, $p(B|F) = 4/5$, (so $p(W|F) = 1/5$), $p(B|T) = 1/2$ (so $p(W|T) = 1/2$). We are told that it was a black ball. The problem is to compute the *a posteriori* probability that Fingers did it, $p(F|B)$. The formula is

$$p(F|B) = \frac{p(B|F)p(F)}{p(B|F)p(F) + p(B|T)p(T)}.$$

Here are the numbers in our case:

$$p(F|B) = \frac{4/5 \times 1/4}{(4/5 \times 1/4) + (1/2 \times 3/4)} = \frac{8}{23},$$

which is the same as $80/230$, i.e., the same as what we got before.